K25U 0321

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Mathematics (Honours) Degree (C.B.C.S.S. – OBE – Regular/Supplementary/Improvement) Examination, April 2025

(2021 and 2022 Admissions)

Core Course

6B24 BMH : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Define level sets.
2. Sketch the vector field on $\mathbb{R}^2$ of X(p) = (p, X(p)) where X(p) = (1, 0).
3. Define n-surface.
4. Define Weingarten map.
5. Define the term Levi-Civita parallel.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Define the velocity vector of a parametrized curve and find the velocity of the parametrized curve $\alpha(t) = (\cos t, \sin t, t)$.
7. Explain why an integral curve of a vector field cannot cross itself as does the parametrized curve.
8. Define smooth functions and smooth vector fields.
9. "Every surface is always open". Is this statement true ? Justify your answer.
10. Let X and Y be smooth vector fields along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$, then prove that $(X \dot{+} Y) = \dot{X} + \dot{Y}$.
11. Define connected set.

P.T.O.

12. Compute $\nabla_v f$ where $f(x_1, x_2) = x_1^2 - x_2^2$, $v = (1, 1, \cos\theta, \sin\theta)$.
13. What is circle of curvature ?
14. Explain the geometric meaning of L_p.

SECTION – C

Answer **any 8** questions. **Each** question carries **4** marks. **(8×4=32)**

15. Find the integral curve through p = (1, 1) of the vector field on $\mathbb{R}^2$ of X(p) = (p, X(p)) where X(p) = (0, 1).
16. Determine whether the vector field $X(x_1, x_2) = (x_1, x_2, 1, 0)$, $U = \mathbb{R}^2$ is complete or not.
17. Find all the level sets of the function $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + x_2^2 + \dots + x_{n+1}{}^2$.
18. Prove that geodesics have constant speed.
19. For each a, b, c, d $\in \mathbb{R}$ prove that the parametrized curve $\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$.
20. Prove that the unit n-sphere is an n-surface.
21. Let $S \subset \mathbb{R}^{n+1}$ be a connected n-surface in $\mathbb{R}^{n+1}$. Then prove that there exist on S exactly two smooth unit normal vector fields N_1 and N_2, and $N_2(p) = -N_1(p)$ for all $p \in S$.
22. Let $f : U \to \mathbb{R}$ be smooth function on U, U is open in $\mathbb{R}^n$. Then prove that the graph of f is an n-surface in $\mathbb{R}^{n+1}$.
23. Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field N. Let $p \in S$ and $v \in S_p$. Then for every parametrized curve $\alpha : I \to S$, with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$, prove that $\ddot{\alpha}(t_0) \cdot N(p) = L_p(v) \cdot v$.
24. Compute the Weingarten map of the hyperplane. $a_1x_1 + a_2x_2 + \dots + a_{n+1}x_{n+1} = b$, where $a = (a_1, a_2, \dots, a_{n+1}) \neq (0, 0, \dots, 0)$.

25. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$ and let α be a piecewise smooth parametrized curve from p to q. Then prove that parallel transport $P_\alpha : S_p \to S_q$ along α is a linear map which is one to one and onto.
26. Let C be the circle $f^{-1}(r^2)$, where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$, oriented by the outward normal vector $\frac{\nabla f}{\|\nabla f\|}$. Prove that the curvature at each point of C is $\frac{-1}{r}$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let c = f(p). Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $(\nabla f(p))^{\perp}$.
28. Let a, b, c $\in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1 and λ_2 are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.
29. State and prove any three properties of Levi-Civita parallelism.
30. Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be a local parametrization of the oriented plane curve C. Show that $K \circ \alpha = \frac{x'y'' - y'x''}{(x^2 + y^2)^{3/2}}$.

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