K25U 0992

Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree
(C.B.C.S.S. – Supplementary) Examination, April 2025
(2019 and 2020 Admissions)
BHM401 : ADVANCED REAL ANALYSIS AND METRIC SPACES

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Define Riemann sum of a function.
2. Define the distance between two nonempty subsets.
3. Find the norm of the partition $P = \{0, 1, 2, 4\}$ of $I = [0, 4]$.
4. Find $\lim \frac{nx}{1+x^2n^2}\ \forall\ x \in \mathbb{R}$.
5. State substitution theorem.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Evaluate $\int_{-10}^{10} \text{sgn}(x)dx$ where sgn(x) denotes the signum function.
7. Differentiate between Pointwise convergence and uniform convergence of sequence of functions.
8. If $f \in R[a, b]$ and if $[c, d] \subseteq [a, b]$ then prove that the restriction of f to [c, d] is in R[c, d].
9. Is an arbitrary union of closed sets closed ? Justify.
10. State Weierstrass M-test.

P.T.O.

11. Let (X, d) be a metric space. Then prove that empty set and the whole space X are closed sets.
12. State Holder's Inequality.
13. State and prove arithmetic mean-geometric mean inequality.
14. Define cantor set.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. If $f \in R[a, b]$ then prove that the value of the integral is uniquely determined.
16. If g is integrable on [a, b] and if f(x) = g(x) except for a finite number of points in [a, b] then prove that f is Riemann integrable and $\int_a^b f = \int_a^b g$.
17. If $f : [a, b] \to \mathbb{R}$ is continuous on [a, b], then prove that $f \in R[a, b]$.
18. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then prove that f is continuous on A.
19. Show that $G_n(x) = x^n(1 - x)$ for $x \in A = [0,1]$ converges uniformly on A.
20. Define Cauchy sequence. Does every Cauchy sequence converges ? Justify.
21. Let $X = C[0,1]$, for $x, y \in X$ define $d(x, y) = \sup\{x(t) - y(t) : 0 \le t \le 1\}$. Calculate the distance between $x(t) = t$ & $y(t) = t^2$.
22. Let (X, d) be a metric space and A, B be subsets of X. Then prove the following :
    a) If $A \subseteq B$ then $A^0 \subseteq B^0$.
    b) $(A \cap B)^0 = A^0 \cap B^0$.

23. If Y is a nonempty subset of a metric space (X, d), and (Y, d_Y) is complete, then prove that Y is closed in X.
24. Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d' is a metric on X.
25. Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.
26. Show that Thomae's function is Riemann integrable on [0,1].

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Let (f_n) be a sequence of functions in R[a, b] and suppose that (f_n) converges uniformly on [a, b] to f. Then prove that $f \in R[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.
28. The space B(S) of all real-or complex-valued functions f on S, each of which is bounded, with the uniform metric $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$ $f, g \in B(S)$. Prove that B(S) with this metric is complete.
29. Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to f(a).
30. Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then prove that $f \in R[a, b]$ if and only if the restrictions to [a, c] and [c, b] are both Riemann integrable.