K25U 0988

Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – O.B.E. – Regular/Supplementary/Improvement) Examination, April 2025
(2021 – 2023 Admissions)

4B15 BMH : INTRODUCTION TO ABSTRACT ALGEBRA AND LINEAR ALGEBRA

Time : 3 Hours　　　　　　　　　　　　　　　　　　　　Max. Marks : 60

SECTION – A

Answer **any four** questions. **Each** question carries **1** mark **each.**　　　　(4×1=4)

1. Does the operation 'ordinary subtraction' is a binary operation on the set of integers $\mathbb{Z}$?
2. Does the group $\mathbb{Z}_6$ is a cyclic group ? Justify your answer.
3. Give an example of a non abelian group.
4. Write the standard ordered basis of $\mathbb{R}^3$.
5. Does the set $\{(1, 1)^T, (2, 2)^T\}$ is linearly independent set in $\mathbb{R}^2$? Justify your answer.

SECTION – B

Answer **any six** questions from the following. **Each** question carries **2** marks.　　(6×2=12)

6. What is the subspace generated by $\{(1, 0, 0, -1)^T, (0, 0, 0, 1)^T\}$ in $\mathbb{R}^4$?
7. State the Rank-Nullity theorem.
8. Does the set $V = \{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$ is a subspace of $\mathbb{R}^2$? Justify your answer.
9. Express the vector $w = (1, 0)^T$ in $\mathbb{R}^2$ as a linear combination of the vectors $v_1 = (1, -1)^T$ and $v_2 = (-1, -1)^T$.
10. Prove or disprove that every group order 6 is abelian.

P.T.O.

11. Let G be a group and $a \in G$. Prove that the order of a is same as the order of a^{-1}.
12. Does the set $\{\sigma \in S_4 : \sigma(1) = 1 \text{ and } \sigma(3) = 3\}$ is a subgroup of S_4 ? Justify your answer.
13. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in S_4$. Find σ^{-21}.
14. Let G be a group and $a, b \in G$. Prove that $(ab)^{-1} = b^{-1} a^{-1}$.

SECTION – C

Answer **any eight** questions. **Each** question carries **4** marks **each.**　　　　(8×4=32)

15. If $ab = a^{-1} b^{-1}$ for all a, b in a group G. Show that G must be abelian.
16. Prove the following : Let G be a group and $a, b \in G$. Then the equations $ax = b$ and $xa = b$ have unique solutions in G.
17. Find all subgroups of the group $\mathbb{Z}_{15}$.
18. Show that the set A_n is a subgroup of S_n.
19. Show that the set of all 2×2 matrices with determinant one is a group under matrix multiplication.
20. Let G be a group and $a, b \in G$. Show that $(aba^{-1})^n = ab\,a^{-1}$ if and only if $b^n = b$.
21. Prove that subgroup of a cyclic group is cyclic.
22. Prove or disprove : If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.
23. Find the null space and dimension of the null space of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
24. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. Find the matrix representation of T with respect to the standard basis $\{(1, 0)^T, (0, 1)^T\}$. Also find T^{-1}.
25. Let $T : V \to W$ be a linear transformation. If $v_1, v_2, \ldots, v_n$ are elements in V such that $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent, show that $v_1, v_2, \ldots, v_n$ are linearly independent.
26. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 2 & -3 & 4 & 5 \\ 0 & 4 & 0 & 8 & 0 \end{pmatrix}$.

SECTION – D

Answer **any two** questions. **Each** question carries **6** marks **each.**　　　　(2×6=12)

27. Find the orbits and cycles of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 9 & 8 \end{pmatrix}$.
28. Prove or disprove the following statements :
    i) The set of rationals $\mathbb{Q}$ is cyclic.
    ii) Every abelian group is cyclic.
    iii) S_n is non abelian for $n > 2$.
29. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x+y+z \end{pmatrix}$.
    i) Verify the Rank-Nullity Theorem for T.
    ii) Is T is invertible ? If so, find T^{-1}.
30. i) Show that every subset of a linearly independent set is linearly independent.
    ii) Let V be the space of all twice differentiable functions on [0, 1]. Find all $x(t) \in V$ such that $x(t)$ and $x'(t)$ are linearly dependent.