K25U 0993

Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree
(C.B.C.S.S. – Supplementary) Examination, April 2025
(2019 and 2020 Admissions)
BHM 402 : ADVANCED ABSTRACT ALGEBRA

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark.

1. Determine whether the map $\phi : \mathbb{R}^* \to \mathbb{R}^*$ under multiplication given by $\phi(x) = |x|$ is a homomorphism.
2. Define index of a subgroup H of a group G.
3. Define simple group.
4. Is the matrix ring $M_2(\mathbb{Z}_2)$ is an integral domain ? Justify.
5. Compute $(x+1)^2$ in $\mathbb{Z}_2[x]$. **(4×1=4)**

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks.

6. Find the partition of $\mathbb{Z}_6$ into cosets of the subgroup $H = \{0, 3\}$.
7. Find the index of $\langle 3 \rangle$ in the group $\mathbb{Z}_{24}$.
8. State the fundamental homomorphism theorem.
9. Let H be a normal subgroup of G. Prove that $\gamma : G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.
10. Find the remainder of 8^{103} when divided by 13.

P.T.O.

11. Find all solutions of the congruence $12x \equiv 27 \pmod{18}$.
12. Let m be a positive integer and let $a \in \mathbb{Z}_m$ be relatively prime to m. For each $b \in \mathbb{Z}_m$, prove that the equation $ax = b$ has a unique solution in $\mathbb{Z}_m$.
13. Let $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g(x) = x^2 - 2x + 3$ in $\mathbb{Z}_5[x]$. Using division algorithm, find q(x) and r (x) so that $f(x) = g(x)\,q(x) + r(x)$.
14. Let $\phi_3 : \mathbb{Z}_7[x] \to \mathbb{Z}_7$ be the evaluation homomorphism. Compute $\phi_3[(x^4 + 2x)(x^3 - 3x^2 + 3)]$. **(6×2=12)**

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks.

15. Let ϕ be a homomorphism from a group G into a group G′. If K′ is a subgroup of G′, then prove that $\phi^{-1}[K']$ is a subgroup of G.
16. Let $\phi : G \to G'$ be a group homomorphism of G onto G′. Prove that if G is abelian then G′ is abelian.
17. Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$ be the mapping given by $\phi(A) = \det A$. Prove that ϕ is a homomorphism and find its kernel.
18. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle(0,1)\rangle$.
19. Let H be a subgroup of G. Prove that H is normal if and only if $(aH)(bH) = (ab)H, \forall a, b \in G$.
20. Let G be a group and C be the commutator subgroup of G. If N is a normal subgroup of G, then prove that G/N is abelian if and only if $C \leq N$.
21. Prove that the cancellation laws hold in a ring R if and only if R has no divisors of 0.
22. Show that for every integer n, the number $n^{33} - n$ is divisible by 15.
23. Prove that every finite integral domain is a field.
24. Factorize the polynomial $x^4 + 3x^3 + 2x + 4$ into linear factors in $\mathbb{Z}_5[x]$.

25. Prove that the polynomial $\phi_p(x) = \dfrac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \dots + x + 1$ is irreducible over $\mathbb{Q}$ for any prime p.
26. State and prove Factor Theorem. **(8×4=32)**

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks.

27. Let ϕ be a homomorphism from a group G into a group G′. Prove that Ker (ϕ) is a normal subgroup of G.
28. Prove that the following are three equivalent conditions for a subgroup H of a group G to be a normal subgroup of G.
    a) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.
    b) $ghg^{-1} = H$ for all $g \in G$.
    c) $gH = Hg$ for all $g \in G$.
29. Prove that the set G_n of non-zero elements of $\mathbb{Z}_n$ that are not zero divisors forms a group under multiplication modulo n.
30. If G is a finite subgroup of the multiplicative group $\langle F^*, \cdot \rangle$ of a field F, then prove that G is cyclic. **(2×6=12)**