Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (C.B.S.S. – Supplementary)
Examination, October 2024
(2021 and 2022 Admissions)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Define countable set. Prove that the set of all integers is countable.

2. If p is a limit point of a set E, then prove that every neighborhood of p contains infinitely many points of E.

3. When can you say that a function f is said to be differentiable at a point x ? Suppose f and g are defined on [a, b] and are differentiable at a point $x \in [a, b]$, then prove that f + g is differentiable at x and $(f + g)'(x) = f'(x) + g'(x)$.

4. Suppose f is differentiable in (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then prove that f is constant.

5. Let f be an increasing function defined on [a, b] and let $x_0, x_1, ..., x_n$ be n + 1 points such that $a = x_0 < x_1 < x_2 < ... < x_n = b$. Then prove that $\sum_{k=1}^{n-1}[f(x_k+) - f(x_k-)] \leq f(b) - f(a)$.

6. Prove that $f(x) = x^2 \cos\left(\frac{1}{x}\right)$, if $x \neq 0, f(0) = 0$ is of bounded variation on [0, 1].

P.T.O.

PART – B

Answer **any four** questions from this part **without** omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit– I

7. a) Let $\{E_n\}$, n = 1, 2, 3, ... be a sequence of countable sets and put $S = \cup_{n=1}^{\infty} E_n$. Then prove that S is countable.

   b) Suppose $y \subset \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

8. a) Show that there exist perfect sets in R^1 which contain no segment.

   b) Prove that a subset E of the real line R^1 is connected if and only if it has the following property.

   If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

9. a) Prove that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

   b) Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then prove that f(X) is compact.

   c) If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then prove that f(E) is connected.

Unit – II

10. a) Suppose f is continuous on [a, b], $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point f(x). If $h(t) = g(f(t))$, $(a \leq t \leq b)$, then prove that h is differentiable at x and $h'(x) = g'(f(x))f'(x)$.

    b) Give an example of a function f, which is differentiable at all points x, but f′ is not a continuous function. Justify.

11. a) State and prove Taylor's theorem.

    b) Suppose f is a continuous mapping of [a, b] into R^k and f is differentiable in (a, b). Then prove that there exists $x \in (a, b)$ such that $|f(b) - f(a)| \leq (b - a)|f'(x)|$.

    c) Define Riemann-Stieltjes integral of f with respect to α over [a, b].

12. a) Suppose f is bounded on [a, b], f has only finitely many points of discontinuity on [a, b], and α is continuous at every point at which f is discontinuous. Then prove that $f \in R(\alpha)$.

    b) Suppose $f \in R(\alpha)$ on [a, b], $m \leq f \leq M$, ϕ is continuous on [m, M] and $h(x) = \phi(f(x))$ on [a, b]. Then prove that $h \in R(\alpha)$ on [a, b].

Unit – III

13. a) State and prove the formula for "integration by parts".

    b) If f and F map [a, b] into R^k, if $f \in R$ on [a, b], and if $F' = f$, then prove that $\int_a^b f(t)dt = F(b) - F(a)$.

    c) If f maps [a, b] into R^k and if $f \in R(\alpha)$ for some monotonically increasing function α on [a, b], then prove that $|f| \in R(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$.

14. a) Let f be of bounded variation on [a, b] and assume that $c \in (a, b)$. Then prove that f is of bounded variation on [a, c] and on [c, b] and $V_f(a, b) = V_f(a, c) + V_f(c, b)$.

    b) Let f be of bounded variation on [a, b]. Let V be defined on [a, b] by $V(x) = V_f(a, x)$ if $a < x \leq b$, $V(a) = 0$. Then prove that

       i) V is an increasing function on [a, b].

       ii) V - f is an increasing function on [a, b].

15. a) Define Rectifiable paths and its arc length. If $c \in (a, b)$ then prove that $\Lambda_f(a, b) = \Lambda_f(a, c) + \Lambda_f(c, b)$.

    b) Consider a rectifiable path f defined [a, b]. If $x \in (a, b]$, let $s(x) = \Lambda_f(a, x)$ and let $s(a) = 0$. Then prove that

       i) The function s so defined is increasing and continuous on [a, b].

       ii) If there is no subinterval of [a, b] on which f is constant, then s is strictly increasing on [a, b].

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