K25U 0331

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Mathematics (Honours) Degree (CBCSS-Supplementary) Examination, April 2025
(2019 – 2020 Admissions)
Core Course
BHM604A : DISCRETE FOURIER ANALYSIS

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions. **Each** question carries **1** mark. **(4×1=4)**

1. Compute the Discrete Fourier Transform (DFT) of the sequence $\{1, 0, -1, 0\}$.
2. Define circulant matrix with an example.
3. What is the conjugate reflection of an element in $l^2(Z_N)$?
4. Define convolution of two sequences in $l^2(Z)$.
5. How is the downsampling operator defined for a sequence in $l^2(Z)$, and what is its effect on the sequence ?

SECTION – B

Answer **any 6** out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Compute the Discrete Fourier Transform (DFT) of $\{1, -1, 1, -1\}$.
7. If X(k) is the DFT of a sequence x(n), express the DFT of x(n –1) in terms of X(k).
8. Verify if the transformation $T(x(n)) = x(n) + x(n-1)$ satisfies linearity.
9. Suppose $z, w \in l^2(Z_N)$. For any $k \in Z$, show that $z * \overline{w}(k) = \langle z, R_k w \rangle$.
10. Suppose $z \in l^2(Z_N)$. Prove that $(U(\overline{z}))(n) = \overline{z}(n)$ for all n.

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11. Verify if the sequence $x(n) = \{1, 1/2, 1/3, ...\}$ belongs to $l^2(Z)$.
12. Suppose $\sum_{n \in Z} w(n)$ be a series of complex numbers which converges absolutely. Prove that the series $\sum_{n \in Z} w(n)$ converges.
13. Find the Fourier coefficients of the function $f : [-\pi, \pi) \to R$ defined as :
$$f(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta < \pi \\ 0, & \text{if } -\pi \le \theta < 0. \end{cases}$$
14. Construct the first-stage wavelet on Z using the filter $h = \{1, -1\}$.

SECTION – C

Answer **any 8** out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Prove that the set $\{E_0, E_1, ..., E_{N-1}\}$ an orthonormal basis for $l^2(Z_N)$, where the functions E_k are defined as $E_m(n) = \frac{1}{\sqrt{N}} e^{i\frac{2\pi mn}{N}}$, m, n = 0, 1, ..., N –1 ?
16. Define $z \in l^2(Z_{512})$ by $z(n) = 3\sin\left(\frac{2\pi 7n}{512}\right) - 4\cos\left(\frac{2\pi 8n}{512}\right)$. Find $\hat{z}$.
17. Define the operator $T : l^2(Z_N) \to l^2(Z_N)$ by $(T(z))(n) = z(n + 1) - z(n)$. Find all eigenvalues of T.
18. Let $z, w \in l^2(Z_N)$. Then, for any $k, j \in Z$, prove that $\langle R_k z, R_j w \rangle = \langle z, R_{j-k} w \rangle = \langle R_{k-j} z, w \rangle$.
19. Suppose N = 2M for some $M \in N$. Define $u, v \in l^2(Z_N)$ by $u = \frac{1}{\sqrt{2}}(1, 1, 0, 0, ..., 0)$, $v = \frac{1}{\sqrt{2}}(1, -1, 0, 0, ..., 0)$. Then prove that the set $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ forms an orthonormal basis for $l^2(Z_N)$.
20. Let $z, w \in l^2(Z_N)$. Then prove that the conjugate reflection of the convolution of z and w satisfies : $(\widetilde{z * w})(k) = \tilde{z}(k) * \tilde{w}(k)$.
21. Prove that the Dirac delta function $\delta(n)$ belongs to $l^2(Z)$ and compute its Fourier transform.

22. Suppose $\{z_k\}_{k=M}^{\infty}$ is a Cauchy sequence in $l^2(Z)$. Prove that for each $n \in Z$, the sequence $\{z_k(n)\}_{k=M}^{\infty}$ is a Cauchy sequence in C.
23. Define the function $f(\theta) = \frac{1}{\sqrt{|\theta|}}$, for $\theta \ne 0$ and $f(0) = 0$. Show that $f \in L^1([-\pi, \pi))$ but $f \notin L^2([-\pi, \pi))$.
24. Suppose $z \in l^2(Z)$ and $w \in l^1(Z)$. Show that $z * w \in l^2(Z)$ and prove the inequality $\| z * w \|_2 \le \| w \|_1 \| z \|_2$.
25. Suppose $z \in l^2(Z)$. Prove that $\left(\widehat{U(z)}\right)(\theta) = \hat{z}(2\theta)$ for all θ.
26. Suppose $w \in l^2(Z)$. If there exist $k, j \in Z$ with $k \ne j$ such that $R_k w = R_j w$, prove that w = 0.

SECTION – D

Answer **any 2** out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Let $T : l^2(Z_N) \to l^2(Z_N)$ be a translation-invariant linear transformation. Then prove that each element of the Fourier basis F is an eigenvector of T. In particular, T is diagonalizable.
28. Suppose $M \in N$, N = 2M and let $u \in l^2(Z_N)$ be such that the set $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(Z_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Then prove that the set $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ forms a first-stage wavelet basis for $l^2(Z_N)$.
29. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that for each $m \in Z$, there exists $\lambda_m \in C$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.
30. Prove that the system matrix $A(\theta)$ associated with the sequences u and v is unitary for all θ, where $u \in l^1(Z)$ and the set $\{R_{2k}u\}_{k \in Z}$ is orthonormal in $l^2(Z)$. The sequence $v \in l^1(Z)$ is defined by $v(k) = (-1)^{k-1}\overline{u(1-k)}$.

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