K24U 0234

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Supplementary/Improvement/One Time Mercy Chance) Examination, April 2024
(2016 to 2020 Admissions)
Core Coure
BHM604A : DISCRETE FOURIER ANALYSIS

Time : 3 Hours Max. Marks : 60

SECTION – A

(Answer **any 4** questions out of **5** questions. **Each** question carries **1** mark.) **(4×1=4)**

1. Define translation invariant linear transformation on $l^2(\mathbb{Z}_N)$.
2. Describe first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.
3. Define upsampling operator.
4. When we say that a linear transformation from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ is translation invariant ?
5. Define a first-stage wavelet system for $l^2(\mathbb{Z})$.

SECTION – B

(Answer **any 6** questions out of **9** questions. **Each** question carries **2** marks.) **(6×2=12)**

6. For any $w \in l^2(\mathbb{Z}_N)$, then prove that $w * \delta = w$.
7. Let z = (1, 1, 0, 2) and w = (i, 0, 1, i) be vectors in $l^2(\mathbb{Z}_4)$, then find the vector $z * w$.
8. Suppose $z, w \in l^2(\mathbb{Z}_N)$, then prove for any $k \in \mathbb{Z}$, that

   $z * \tilde{w}\,(k) = \langle z, R_k w\rangle$ and

   $z * w\,(k) = \langle z, R_k \tilde{w}\rangle$.

P.T.O.

9. Suppose $M \in \mathbb{N}$ and N = 2M. Let $u, v \in l^2(\mathbb{Z}_N)$, then $B = \{R_{2K}v\}_{k=0}^{M-1} \cup \{R_{2K}u\}_{k=0}^{M-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if the system matrix A(n) of u and v is unitary for each n = 0, 1, ..., M – 1.
10. Suppose N is divisible by 2^p. Suppose $u, v \in l^2(\mathbb{Z}_N)$ are such that the system matrix A(n) is unitary for all n. Let $u_1 = u$ and $v_1 = v$, and, for all l = 2, 3, ..., p, define u_l by equation $u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1(n + kN/2^{l-1})$ and v_l similarly with v_1 in place of u_1, then $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ is p^{th}-stage wavelet filter sequence.
11. Prove that $L^2([-\pi, \pi))$ is vector space.
12. Prove that trignometric system is complete in $L^2([-\pi, \pi))$.
13. Suppose $M \in \mathbb{N}$ and N = 2M. Suppose $u \in l^1(\mathbb{Z})$, then prove that $\hat{u}_{(N)}(m) = \hat{u}(-2\pi m/N)$.
14. Suppose $w \in l^1(\mathbb{Z})$, then prove that $(w * \tilde{w}) + (w * \tilde{w})^* = 2\delta$.

SECTION – C

(Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks). **(8×4=32)**

15. Define $E_0, E_1, \ldots, E_{N-1} \in l^2(\mathbb{Z}_N)$ by $E_m(n) = \frac{1}{\sqrt{N}} e^{2\pi i mn/N}$ for $0 \le m, n \le N - 1$, then the set $\{E_0, E_1, \ldots, E_{N-1}\}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$.
16. Suppose $z, w \in l^2(\mathbb{Z}_N)$, then prove that for each m, $(z * w)\hat{}(m) = \hat{z}(m)\hat{w}(m)$.
17. Define $T : l^2(\mathbb{Z}_4) \to l^2(\mathbb{Z}_4)$ by T(z)(n) = z(n) + 2z(n + 1) + z(n + 3). Find the eigenvalues and eigenvectors of T, and diagonalize the matrix A representing T in the standard basis, if possible.
18. Suppose $M \in \mathbb{N}$, N = 2M, and $u \in l^2(\mathbb{Z}_N)$ is such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(\mathbb{Z}_N)$ by $v(k)=(-1)^{k-1}\overline{u(1-k)}$ for all k, then prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

19. Suppose N is even, say N = 2M, $z \in l^2(\mathbb{Z}_N)$ and $x, y, w \in l^2(\mathbb{Z}_{N/2})$, then prove that $D(z) * w = D(z * U(w))$ and $U(x) * U(y) = U(x * y)$.
20. Suppose N is divisible by 2, and $u_1 \in l^2(\mathbb{Z}_N)$. Define $u_2 \in l^2(\mathbb{Z}_{N/2})$ by $u_2(n) = u_1(n) + u_1(n + N/2)$, then for all m $\hat{u}_2(m) = \hat{u}_1(2m)$.
21. Prove that $l^2(\mathbb{Z})$ is complete.
22. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that, for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.
23. Suppose $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a bounded, translation-invariant linear transformation. Define $b \in l^2(\mathbb{Z})$ by $b = T(\delta)$. Then prove that for all $z \in l^2(\mathbb{Z})$, $T(z) = b * z$.
24. Suppose $u \in l^1(\mathbb{Z})$ and $\{R_{2k}u\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Define a sequence $v \in l^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$. Then prove that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a complete orthonormal system in $l^2(\mathbb{Z})$.
25. Suppose $M \in \mathbb{N}$ and N = 2M. Suppose $u, v \in l^1(\mathbb{Z})$ are such that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first-stage wavelet system for $l^2(\mathbb{Z})$. Define $u_{(N)}, v_{(N)} \in l^2(\mathbb{Z}_N)$ by $u_{(N)}(n) = \sum_{k\in\mathbb{Z}} u(n + kN)$ and $v_{(N)}(n) = \sum_{k\in\mathbb{Z}} v(n + kN)$. Then prove that $\{R_{2k}v_{(N)}\}_{k=0}^{M-1} \cup \{R_{2k}u_{(N)}\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.
26. Let $p \in \mathbb{N}$, for l = 1, 2, ..., p, suppose that $u_l, v_l \in l^1(\mathbb{Z})$, and the system matrix $A_l(\theta)$ defined as $A_l(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_l(\theta) & \hat{v}_l(\theta) \\ \hat{u}_l(\theta+\pi) & \hat{v}_l(\theta+\pi) \end{bmatrix}$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = v_1$, $g_1 = u_1$, and, inductively, for l = 2, 3, ..., p, define f_l and g_l by $f_l = g_{l-1} * U^{l-1}(v_l)$, $g_l = g_{l-1} * U^{l-1}(u_l)$. Define $B = \{R_{2^l k} f_l : k \in \mathbb{Z}, l = 1, 2, \ldots, p\} \cup \{R_{2^p k} g_p : k \in \mathbb{Z}\}$, then prove that B is a complete orthonormal set for $l^2(\mathbb{Z})$.

SECTION – D

(Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks.) **(2×6=12)**

27. Suppose $z \in l^2(\mathbb{Z}_N)$ and $k \in Z$, then prove that for any $m \in Z$, $(R_k z)\hat{}(m) = e^{-2\pi i mk/N}\hat{z}(m)$.
28. Prove that : Suppose $M \in \mathbb{N}$ and N = 2M. Let $u, v \in l^2(\mathbb{Z}_N)$, then $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if the system matrix A(n) of u and v is unitary for each n = 0, 1,...,M – 1. Equivalently, B is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$ if and only if
    a) $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
    b) $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
    c) $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0$ for all n = 0, 1, . . . , M – 1.
29. Suppose $z \in l^2(\mathbb{Z})$ and $w \in l^1(\mathbb{Z})$, then prove that $z * w \in l^2(\mathbb{Z})$ and $\|z * w\| \le \|w\|_1 \|z\|$.
30. Suppose l is a positive integer, $g_{l-1} \in l^2(\mathbb{Z})$, and $\{R_{2^{l-1}k}\, g_{l-1}\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Suppose also that $u, v \in l^1(\mathbb{Z})$ and the system matrix A(θ) of u and v is unitary for all θ. Define $f_l = g_{l-1} * U^{l-1}(v)$ and $g_l = g_{l-1} * U^{l-1}(u)$, then $\{R_{2^l k} f_l\}_{k\in\mathbb{Z}} \cup \{R_{2^l k} g_l\}_{k\in\mathbb{Z}}$ is orthonormal.
