K24U 0226

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Honours in Mathematics Degree (CBCSS – OBE – Regular) Examination, April 2024
(2021 Admissions)
Core Course
6B26 BMH : MEASURE THEORY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any four** questions out of the five questions. **Each** question carries **1** mark. **(4×1=4)**

1. Find lim inf x_n, where $(x_n) = \left(\frac{-1}{n}\right)$.
2. Give an example of a σ-algebra on $\mathbb{R}$, the set of real numbers.
3. Define the counting measure on $\mathbb{N}$, the set of natural numbers.
4. Define the integral of a simple function in $M^+(X, \mathbf{X})$.
5. If f is an X-measurable real-valued function and if f(x) = 0 for μ-almost everywhere, then find $\int f d\mu$.

SECTION – B

Answer **any six** questions out of the nine questions. **Each** question carries **2** marks. **(6×2=12)**

6. Prove that the characteristic function χ_E is measurable.
7. Let **B** be the Borel Algebra on $\mathbb{R}$, the set of real numbers. Show that any continuous function $f : \mathbb{R} \to \mathbb{R}$ is **B**-measurable.
8. Let μ be a measure on a σ-algebra **X** and A be a fixed set in X. Show that the function λ defined for $E \in \mathbf{X}$ by $\lambda(E) = \mu(A \cap E)$ is a measure on X.

P.T.O.

9. Let μ be a measure on a σ-algebra **X**. If $E, F \in \mathbf{X}$ with $E \subset F$, then prove that $\mu(E) \leq \mu(F)$.
10. By using properties of outer measure, show that the interval [0, 1] is not countable.
11. Prove that the translate of a Lebesgue measurable set is Lebesgue measurable.
12. Is the empty set Lebesgue measurable ? Justify your answer.
13. If $f \in L(X, \mathbf{X}, \mu)$ and g is an **X**-measurable real-valued function with f(x) = g(x) almost everywhere on X, then prove that $g \in L(X, \mathbf{X}, \mu)$ and $\int f d\mu = \int g d\mu$.
14. If $f \in L(X, \mathbf{X}, \mu)$ and $\lambda : \mathbf{X} \to \mathbb{R}$ is defined by $\lambda(E) = \int_E f d\mu$, then prove that λ is a charge.

SECTION – C

Answer **any eight** questions out of the twelve questions. **Each** question carries **4** marks. **(8×4=32)**

15. Prove that an extended real-valued function f is measurable if and only if the sets $A = \{x \in X : f(x) = +\infty\}$, $B = \{x \in X : f(x) = -\infty\}$ belong to **X** and the real valued function f_1 defined by $f_1(x) = \begin{cases} f(x), & \text{if } x \notin A \cup B \\ 0, & \text{if } x \in A \cup B \end{cases}$ is measurable.
16. If f is a non-negative function in $M(X, \mathbf{X})$, then prove that there exist a sequence (ϕ_n) in $M(X, \mathbf{X})$ such that, $0 \leq \phi_n(x) \leq \phi_{n+1}(x)$, $f(x) = \lim \phi_n(x)$ for $x \in X$, $n \in \mathbb{N}$ and each ϕ_n has only a finite number of real values.
17. Let (f_n) be a sequence in $M(X, \mathbf{X})$. Consider the functions $f(x) = \inf f_n(x)$ and $f^*(x) = \lim\inf f_n(x)$. Prove that f and f* belong to $M(X, \mathbf{X})$.
18. Let μ be a measure defined on a σ-algebra **X**. If (E_n) is an increasing sequence in **X**, then prove that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.
19. Prove that outer measure is countably subadditive.
20. Prove that the union of a countable collection of measurable sets is measurable.
21. State and prove Fatou's Lemma.

22. Let f belong to M^+. Prove that f(x) = 0 μ-almost everywhere on X if and only if $\int f d\mu = 0$.
23. Let ϕ be a simple function in $M^+(X, \mathbf{X})$ and λ be defined for $E \in \mathbf{X}$ by $\lambda(E) = \int \phi \chi_E \, d\mu$. Prove that λ is a measure on **X**.
24. State and prove the property of absolute integrability of the Lebesgue Integral.
25. For two functions f and g in L, prove that f + g belongs to L and $\int (f + g) d\mu = \int f d\mu + \int g d\mu$.
26. Prove that a measurable function f belongs to L if and only if |f| belongs to L. Further prove that $\left|\int f d\mu\right| \leq \int |f| d\mu$.

SECTION – D

Answer **any two** questions out of the four questions. **Each** question carries **6** marks. **(2×6=12)**

27. Let f, g be measurable real-valued functions and let c be a real number. Then prove that the functions cf, f^2, f + g and fg are also measurable.
28. a) Prove that any set of outer measure zero is measurable.

    b) Let A be any set and $\{E_k\}_{k=1}^{n}$ a finite disjoint collection of measurable sets. Prove that $m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$.
29. State and prove the Monotone Convergence Theorem.
30. State and prove Lebesgue Dominated Convergence Theorem.
