K24U 0224

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Mathematics (Honours) Degree (C.B.C.S.S. – OBE – Regular) Examination, April 2024
(2021 Admissions)
Core Course
6B24BMH : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 60

I. Answer **any 4** questions. **Each** question carries **1** mark. **(4×1=4)**

1) Define graph of a function.

2) Find the gradient field of the function $f(x_1, x_2) = x_1^2 + x_2^2$.

3) Define geodesic.

4) Define covariant derivative.

5) Define curvature.

II. Answer **any 6** questions. **Each** question carries **2** marks. **(6×2=12)**

1) Sketch typical level curves and the graph of the function $f(x_1, x_2) = x_1 - x_2$.

2) Define divergence of a vector field and find the divergence of the vector field $X(x_1, x_2) = \left(x_1, x_2, \frac{x_1}{2}, \frac{x_2}{2}\right)$.

3) Prove that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.

4) For what values of c is the level set $f^{-1}(c)$ an n–surface, where $f(x_1, x_2 ...., x_{n+1}) = x_1^2 + ... + x_n^2 - x_{n+1}^2$.

5) Show that the unit n-sphere is connected if $n > 1$.

6) Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.

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7) Prove that the value of the derivative of f with respect to v is independent of the choice of α.

8) Compute $\nabla_v f$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $v \in \mathbb{R}_p^{n+1}, p \in \mathbb{R}^{n+1}$, are given by $f(x_1, x_2) = 2x_1^2 + 3x_2^2$, $v = (1, 0, 2, 1)$, $(n = 1)$.

9) Compute $\nabla_v X$ where $v \in \mathbb{R}_p^{n+1}, p \in \mathbb{R}^{n+1}$ and X are given by $X(x_1, x_2) = (x_1, x_2, -x_2, x_1)$, $v = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$, $(n = 1)$.

III. Answer **any 8** questions. **Each** question carries **4** marks. **(8×4=32)**

1) Define the following :
   a) Gradient of a smooth function.
   b) Parameterized curve.
   c) Velocity of a parameterized curve.
   d) Integral curve.

2) Let U be an open set in $\mathbb{R}^{n+1}$, let $p \in U$ and let X be a smooth vector field on U. Let $\alpha : I \to U$ be the maximal integral curve of X through p. Show that if $\beta : \bar{I} \to U$ is any integral curve of X, with $\beta(t_0) = p$ for some $t_0 \in \bar{I}$, then $\beta(t) = \alpha(t - t_0)$ for all $t \in \bar{I}$.

3) Find the integral curve through $p = (a, b)$ of the vector field $X(p) = (p, X(p))$ where $X(-x_2, x_1)$.

4) Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$ and let X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then prove that $\alpha(t) \in S$ for all $t \in I$.

5) Show that if S is a connected n-surface in $\mathbb{R}^{n+1}$ and $g : S \to \mathbb{R}$ is smooth and takes on only the values + 1 and –1, then g is constant.

6) Verify the following properties of the differentiation of vector fields along parametrized curves :
   a) $\dot{\overline{X+Y}} = \dot{X} + \dot{Y}$
   b) $(X\dot{Y})' = \dot{X}.Y + X.\dot{Y}$,

   Where X + Y and X.Y are defined along α by $(X + Y)(t) = X(t) + Y(t)$, $(X.Y)(t) = X(t).Y(t)$, $\forall t \in I$.

7) Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (\cos 3t, \sin 3t)$.

8) Let S be an n-plane $a_1x_1 + ... + a_{n+1}x_{n+1} = b$ in $\mathbb{R}^{n+1}$, let $p, q \in S$ and let $v = (p, v) \in S_p$. Show that if α is any parametrized curve in S from p to q then $P_\alpha(v) = (q, v)$. Conclude that, in an n-plane, parallel transport is path independent.

9) Prove that Weingarten map of the n-sphere of radius r is simply multiplication by 1/r.

10) Prove that local parametrizations of plane curves are unique up to reparametrization.

11) Find the curvature k of the oriented plane curve $x_2 = ax_1^2 = c, a \neq 0$.

IV. Answer **any 2** questions. **Each** question carries **6** marks. **(2×6=12)**

1) Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.

2) State and prove the existence and uniqueness theorem for integral curves for smooth vector fields.

3) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let to $t_0 \in I$ and let $v \in S_{\alpha(t_0)}$. Then prove that there exists a unique vector field V, tangent to S along α, which is parallel and has $V(t_0) = v$.

4) Prove that the Weingarten map L_p is self-adjoint; that is, $L_p(v).w = v.L_p(w)$ for all $v, w \in S_p$.
