Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Supplementary/One Time Mercy Chance) Examination, April 2024
(2016 to 2020 Admissions)
BHM 402 : ADVANCED ABSTRACT ALGEBRA

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark.

1. Define normal subgroup of a group G.
2. Let ϕ be a homomorphism from a group G into a group G′. If e is the identity element in G then prove that $\phi(e)$ is the identity element in G′.
3. True or false : The alternating group A_4 is simple.
4. Check whether $25x^5 - 9x^4 - 3x^2 - 12$ is irreducible over $\mathbb{Q}$.
5. Find the units in $\mathbb{Z}_{14}$. **(4×1=4)**

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks.

6. Prove that every group of prime order is cyclic.
7. Prove that a group homomorphism $\phi : G \to G'$ is a one-to-one map if and only if Ker $(\phi) = \{e\}$.
8. Prove that every subgroup of an abelian group is normal.
9. Let n be a positive integer. Compute $\frac{\mathbb{R}}{n\mathbb{R}}$.

P.T.O.

10. Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$.
11. Let R be a ring with additive identity 0. Then for $a \in R$, prove that $0a = a0 = 0$.
12. Is the matrix ring $M_2(\mathbb{Z}_2)$ is an integral domain ? Justify.
13. Prove that $a \in F$ is a zero of $f(x) \in F[x]$ if and only if $x - a$ is a factor of f(x) in F[x].
14. Let $f(x) = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g(x) = x^2 - 2x + 3$ in $\mathbb{Z}_5[x]$. Using division algorithm, find q(x) and r(x) so that $f(x) = g(x)q(x) + r(x)$. **(6×2=12)**

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks.

15. Find all left cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$.
16. Let S_n be the symmetric group on n letters, and let $\phi : S_n \to \mathbb{Z}_2$ be defined by $\phi(\sigma) = \begin{cases} 0, & \text{if } \sigma \text{ is an even permutation} \\ 1, & \text{if } \sigma \text{ is an odd permutation} \end{cases}$. Show that ϕ is a homomorphism.
17. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2, 3)\rangle$.
18. Let H be a normal subgroup of G. Prove that G/H is a group under the binary operation (aH) (bH) = (ab) H.
19. Prove that M is a maximal normal subgroup of G if and only if G/M is simple.
20. Let G be a group and C be the commutator subgroup of G. If N is a normal subgroup of G then prove that G/N is abelian if and only if $C \leq N$.
21. Prove that in the ring $\mathbb{Z}_n$ the divisors of 0 are precisely those non-zero elements that are not relatively prime to n.
22. State and prove Little Theorem of Fermat.
23. Find all solutions of the congruence $15x \equiv 27 \pmod{18}$.
24. Factorize the polynomial $x^4 + 4$ into linear factors in $\mathbb{Z}_5[x]$.
25. Show that $f(x) = x^4 - 2x^2 + 8x + 1 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.
26. Prove that for a field F, the factorization in F[x] is unique. **(8×4=32)**

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks.

27. Let H be a subgroup of G. Prove that the relation '~' defined by a ~ b if and only if $a^{-1}b \in H$ is an equivalence relation on G.
28. Prove the falsity of the converse of the theorem of Lagrange.
29. Prove that
    a) Every finite integral domain is a field.
    b) If p is a prime, then $\mathbb{Z}_p$ is a field.
30. State and prove Eisenstein's Criterion. **(2×6=12)**