K24U 3610

Reg. No. : ....

Name : ....

III Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Supplementary) Examination, November 2024
(2019 and 2020 Admissions)
BHM 305 : ADVANCED LINEAR ALGEBRA

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Let F be a field and let f be the linear functional on F^2 defined by $f(x_1, x_2) = ax_1 + bx_2$. Let T be a linear operator on F^2 defined by $T(x_1, x_2) = (x_1, 0)$ and let $g = T^t f$. Find $g(x_1, x_2)$.

2. Find the minimal polynomial for T whose matrix representation is $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$.

3. Let(|) be the standard inner product on R^2. Let $\alpha = (1, 2)$, $\beta = (-1, 1)$. If γ is a vector such that $(\alpha | \gamma) = -1$ and $(\beta | \gamma) = 3$, find γ.

4. Let V be the space C^2, with standard inner product. Let T be the linear operator defined by $T \epsilon_1 = (1, -2)$, $T \epsilon_2 = (i, -1)$. If $\alpha = (x_1, x_2)$, find $T^*\alpha$.

5. Show that $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Let V and W be finite dimensional vector spaces over the field F, and let T be a linear transformation from V into W. Prove that rank (T^t) = rank (T).

7. Find the characteristic values of $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$.

P.T.O.

8. Let T be a linear operator on a finite dimensional space V. Suppose there exists k distinct scalars $c_1, \ldots, c_k$ and k nonzero linear operators $E_1, \ldots, E_k$ which satisfies the following conditions :
   i) $T = c_1E_1 + \ldots + c_kE_k$;
   ii) $I = E_1 + \ldots + E_k$;
   iii) $E_iE_j = 0$, $i \neq j$

   Prove that $E_i^2 = E_i$ and the range of E_i is the characteristic space for T associated with c_i.

9. Let V be a vector space and (|) an inner product on V. Show that
   i) $(0|\beta) = 0$ for all β in V.
   ii) if $(\alpha|\beta) = 0$ for all β in V, then $\alpha = 0$.

10. Let V be an inner product space. Prove that $\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$ for all $\alpha, \beta \in V$.

11. Prove that every finite dimensional inner product space has an orthonormal basis.

12. Let V be a finite dimensional inner product space, and f a linear functional on V. Prove that there exist a unique vector β in V such that $f(\alpha) = (\alpha|\beta)$ for all α in V.

13. Let V be a finite dimensional inner product space, and let T be a linear operator on V. In any orthonormal basis for V, prove that the matrix of T^* is the conjugate transpose of the matrix of T.

14. Let V be an inner product space and T a self adjoint linear operator on V. Prove that each characteristic value of T is real and characteristic vector of T associated with distinct characteristic values are orthogonal.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Let V and W be finite dimensional vector spaces over the field F. Let β be an ordered basis for V with dual basis β^*, and let β' be an ordered basis for W with dual basis β'^*. Let T be a linear transformation from V into W; let A be the matrix of T relative to β, β' and let B be the matrix of T^t relative to β'^*, β'. Prove that $B_{ij} = A_{ji}$.

16. Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \ldots (x - c_k)$ where $c_1, \ldots, c_k$ are distinct elements of F.

17. Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis of R^3, each vector of which is a characteristic vector of T.

18. Find an invertible real matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -8 \\ 0 & -1 \end{bmatrix}$.

19. Let T be a linear operator on the space V, and let $W_1, \ldots, W_k$ and $E_1, \ldots, E_k$ be on V such that
    i) $E_i^2 = E_i$ for each i;
    ii) $E_iE_j = 0$ if $i \neq j$;
    iii) $I = E_1 + \ldots E_k$;
    iv) the range of E_i is W_i.

    Prove that each subspace W_i is invariant under T if and only if $TE_i = E_iT$ for $i = 1, \ldots, k$.

20. Let V be a vector space of all continuous real valued functions on [0, 1]. Let $f, g \in V$. Define $(f | g)$ by $(f | g) = \int_0^1 f(t)g(t)\,dt$. Prove that (f|g) is an inner product on V.

21. State and prove parallelogram law.

22. State and prove Cauchy Schwarz inequality.

23. Let W be a finite dimensional subspace of an inner product space V and let E be an orthogonal projection of V on W. Then prove that E is an idempotent linear transformation of V onto W, $W^\perp$ is the null space of E and $V = W \oplus W^\perp$.

24. Let V be a finite dimensional inner product space. If T and U are linear operators on V and c is a scalar, prove that
    i) $(T + U)^* = T^* + U^*$

    ii) $(TU)^* = U^*T^*$
    iii) $(T^*)^* = T$

25. On a finite dimensional inner product space of positive dimension, prove that every self adjoint operator has a characteristic vector.

26. Prove that T is normal if and only if $T = T_1 + iT_2$ where T_1 and T_2 are self adjoint operators which commute.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. State and prove Cayley Hamilton theorem.

28. State and prove primary decomposition theorem.

29. State and prove Gram Schmidt orthogonalization process.

30. Prove that for every invertible complex n × n matrix B there exists a unique lower triangular matrix M with positive entries on the main diagonal such that MB is unitary.