Reg. No. : ....

Name : ....

VI Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Regular/Supplementary/Improvement) Examination, April 2023
(2016 Syllabus)
BHM604A : DISCRETE FOURIER ANALYSIS

Time : 3 Hours — Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark : **(4×1=4)**

1. Construct an orthonormal basis for $l^2(\mathbb{Z}_4)$.
2. Define a translation invariant transformation on $l^2(\mathbb{Z})$ and give an example.
3. Define p^{th} stage wavelet filter sequence.
4. Define the downsampling and upsampling operators on $l^2(\mathbb{Z})$.
5. Define a homogeneous wavelet system for $l^2(\mathbb{Z})$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks : **(6×2=12)**

6. If $z \in l^2(\mathbb{Z}_N)$, prove that $\|z\|^2 = (1/N)\|\hat{z}\|^2$.
7. Suppose $z, w \in l^2(\mathbb{Z}_N)$. For any $k \in \mathbb{Z}$ prove that $z * \tilde{w}(k) = \langle z, R_k w\rangle$ and $z * w(k) = \langle z, R_k \tilde{w}\rangle$.
8. Suppose N is divisible by 2^p. Suppose $u, v \in l^2(\mathbb{Z}_N)$ are such that the system matrix A(n) is unitary for all n. Let $u_1 = u$ and $v_1 = v$, and, for all $l = 2, 3, \dots p$, define u_l by the equation $u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1\left(n+\frac{kN}{2^{l-1}}\right)$ and v_l similarly with v_1 in place of u_1. Prove that $u_1, v_1, u_2, v_2, \dots, u_p, v_p$ is a p^{th}-stage wavelet filter sequence.

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9. Define $L^2([-\pi, \pi])$ and prove that it an inner product space.
10. Prove that $l^2(\mathbb{Z})$ is complete.
11. Find out D(z) and U(z) if $z = ((i^n))_{n\in\mathbb{Z}}$. Also prove that $D \circ U(z) = z$ for $z = (z(n))_{n\in\mathbb{Z}}$.
12. Show that the trigonometric system $\{e^{in\theta}\}$ is an orthonormal set in $L^2([-\pi, \pi])$.
13. Suppose $w, z \in l^1(\mathbb{Z})$. Prove that the set $\{R_{2k}w\}_{k\in Z}$ is orthonormal if and only if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta+\pi)|^2 = 2$ for all $\theta \in [0, \pi)$.
14. Suppose $u \in l^1(\mathbb{Z})$ and $\{R_{2k}u\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Define a sequence $v \in l^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$. Then prove that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first-stage wavelet system in $l^2(\mathbb{Z})$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks : **(8×4=32)**

15. Suppose $z \in l^2(\mathbb{Z}_N)$. Prove that z is real (i.e. every component of z is a real number) if and only if $\hat{z}(m) = \overline{\hat{z}(N-m)}$ for all m.
16. Suppose $z, w \in l^2(\mathbb{Z}_N)$. Prove that $(z * w)^\wedge(m) = \hat{z}(m)\hat{w}(m)$ for each m.
17. Find the eigenvalues and eigenvectors of $T : l^2(\mathbb{Z}_4) \to l^2(\mathbb{Z}_4)$ given by $T(z)(n) = z(n) + 2z(n+1) + z(n+3)$.
18. Let $w \in l^2(\mathbb{Z}_N)$. Prove that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis of $l^2(\mathbb{Z}_N)$ if and only if $|\hat{w}(n)| = 1$ for all $n \in \mathbb{Z}_N$.
19. Suppose $M \in \mathbb{N}$, $N = 2M$ and $u \in l^2(\mathbb{Z}_N)$ is such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(\mathbb{Z}_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Then prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

20. Suppose $N = 2^n$, $1 \le p \le n$ and $u_1, v_1, u_2, v_2, \dots, u_p, v_p$ form a p^{th} stage wavelet filter sequence. Suppose $z \in l^2(\mathbb{Z}_N)$. Prove that the output $\{x_1, x_2, x_3, \dots, x_p, y_p\}$ of the analysis phase of the corresponding p^{th} stage wavelet filter bank can be computed using no more than $4N + N\log_2 N$ complex multiplications.
21. Suppose $\hat{u} = (\sqrt{2}, \sqrt{2}, 0, 0)$ and $\hat{v} = (0, 0, \sqrt{2}, \sqrt{2})$. Compute u and v using IDFT and prove that $\{v, R_2v, u, R_2u\}$ is an orthonormal basis for $l^2(\mathbb{Z}_4)$.
22. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded translation- invariant linear transformation. Then prove that for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.
23. Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = 0$ for all $n \in \mathbb{Z}$. Then prove that $f(\theta) = 0$ a.e.
24. Suppose $z \in l^2(\mathbb{Z})$ and $w \in l^1(\mathbb{Z})$. Then prove that $z * w \in l^2(\mathbb{Z})$ and $\|z * w\| \le \|w\|_1 \|z\|$.
25. Suppose $b \in l^1(Z)$. For $z \in l^2(\mathbb{Z})$, define $T_b(z) = b * z$. Prove that T_b is bounded and translation invariant on $l^2(\mathbb{Z})$.
26. Let $p \in \mathbb{N}$. For $l = 1, 2, \dots, p$, suppose $u_l, v_l \in l^1(\mathbb{Z})$ and the system matrix
$$A_l(\theta) = 1/\sqrt{2}\begin{bmatrix} \hat{u}_l(\theta) & \hat{v}_l(\theta) \\ \hat{u}_l(\theta+\pi) & \hat{v}_l(\theta+\pi) \end{bmatrix}$$
is unitary for all $\theta \in [0, \pi)$. Define $f_1 = v_1$, $g_1 = u_1$ and inductively for $l = 2, 3, \dots, p$, $f_l = g_{l-1} * U^{l-1}(v_l)$, $g_l = g_{l-1} * U^{l-1}(u_l)$. Let $B = \{R_{2^l k} f_l : k \in \mathbb{Z}, l = 1, 2, \dots, p\} \cup \{R_{2^p k} g_p : k \in \mathbb{Z}\}$. Then, prove that B is a complete orthonormal set for $l^2(\mathbb{Z})$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks : **(2×6=12)**

27. Let $T : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ be a translation-invariant linear transformation. Then, prove that each element of the Fourier basis F is an eigenvector of T and, in particular, T is diagonalisable.
28. Suppose $M \in \mathbb{N}$ and $N = 2M$. Let $u, v \in l^2(\mathbb{Z}_N)$. Then, prove that :
    a) $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1} = \{v, R_2v, R_4v, \dots, R_{N-2}v, u, R_2u, R_4u, \dots, R_{N-2}u\}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if the system matrix A(n) of u and v is unitary for each $n = 0, 1, 2, \dots, M-1$.
    b) Equivalently, B is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$ if and only if $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$, $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$ and $\hat{u}(n)\overline{\hat{v}(n)} + \hat{u}(n+M)\overline{\hat{v}(n+M)} = 0$ for all $n = 0, 1, \dots, M-1$.
29. Suppose $f : [-\pi, \pi) \to \mathbb{C}$ is continuous and bounded say $|f(\theta)| \le M$ for all θ. If $\langle f, e^{in\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = 0$ for all $n \in \mathbb{Z}$, then prove that $f(\theta) = 0$ for all $\theta \in [-\pi, \pi)$.
30. Suppose l is a positive integer, $g_{l-1} \in l^2(\mathbb{Z})$ and $\{R_{2^{l-1}k}g_{l-1}\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Suppose $u, v \in l^1(\mathbb{Z})$ and system matrix $A(\theta)$ of u and v is unitary for all θ. Define $f_l = g_{l-1} * U^{l-1}(v)$, $g_l = g_{l-1} * U^{l-1}(u)$. Then prove that $\{R_{2^l k}f_l : k \in \mathbb{Z}\} \cup \{R_{2^l k}g_l : k \in \mathbb{Z}\}$ is orthonormal.