Reg. No. : ....

Name : ....

V Semester B.Sc. Honours in Mathematics Degree
(CBCSS – Supplementary/Improvement) Examination, November 2023
(2018 – 2020 Admissions)
BHM 502 : ADVANCED COMPLEX ANALYSIS

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Evaluate $\int_0^1 (1+it)^2\,dt$.
2. State fundamental theorem of algebra.
3. Write the Maclaurin series expansion of $f(z) = e^z$.
4. Find the singular points of $f(z) = \frac{z+1}{z^3(z^2+1)}$.
5. State Rouche's theorem.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Find the value of the integral $\int_C \bar{z}\,dz$, where C is $z = 2e^{i\theta}$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$.
7. Without evaluating the integral, show that $\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6\pi}{7}$, where C is $|z| = 2$.
8. State Cauchy Goursat theorem and use this theorem to $\int_C f(z)\,dz$, where $f(z) = ze^{-z}$ and the contour C is $|z| = 1$.
9. Prove that every absolute convergent series of complex numbers is convergent.
10. State Taylor's theorem.

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11. Show that $\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$, when $0 < |z| < 4$.
12. Show that $\int_C \exp\left(\frac{1}{z^2}\right)dz = 0$, where C is positively oriented unit circle.
13. Find the residue at $z = 0$ of the function $f(z) = \frac{1}{z + z^2}$.
14. State Jordan's lemma.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. State and prove Cauchy integral formula.
16. State and prove Liouville's theorem.
17. Let $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$ and i, the orientation of C being in the counterclockwise direction. Use parametric representation for C or legs of C to evaluate $\int_C f(z)\,dz$.
18. Suppose that $z_n = x_n + iy_n$, $n = 1, 2, \ldots$ and $z = x + iy$. Prove that $\lim_{n\to\infty} z_n = z$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.
19. Obtain the Taylor's series expansion of cos z about $z = \frac{\pi}{2}$.
20. If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $z = z_1$ $(z_1 \neq z_0)$, prove that it is absolutely convergent at each point z in the open disc $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$.
21. State and prove Cauchy residue theorem.
22. Evaluate $\int_C \frac{5z-2}{z(z-1)}\,dz$, where C is the circle $|z| = 2$ in the positive sense.
23. Let f be a function analytic at a point z_0. Prove that f has a zero of order m at z_0 if and only if there is a function g, which is analytic and non-zero at z_0 such that $f(z) = (z - z_0)^m\, g(z)$.
24. Use residues to evaluate $\int_0^{\infty} \frac{dx}{x^2+1}$.

25. Suppose that
    i) a function f(z) has a simple pole at a point $z = x_0$ on the real axis, with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ and with residue B_0;
    ii) C_ρ denotes the upper half of a circle $|z - x_0| = \rho$, where $\rho < R_2$ and clockwise direction is taken.

    Prove that $\lim_{\rho\to 0} \int_{C_\rho} f(z)\,dz = -B_0\pi i$.
26. Evaluate $\int_0^{\infty} \frac{d\theta}{5 + 4\sin\theta}$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. State and prove maximum modulus principle.
28. Find all Laurent series of $f(z) = \frac{-1}{(z-1)(z-2)}$ about $z = 0$.
29. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to positively oriented simple closed contour C, then prove that $\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$.
30. Use residues to find the Cauchy principal values of the improper integral $\int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2 + 2x + 2}$.