K23U 2645

Reg. No. : ....

Name : ....

V Semester B.Sc. Honours in Mathematics Degree (CBCSS – Supplementary/ Improvement) Examination, November 2023
(2018 – 2020 Admissions)
BHM505 A : INTEGRAL EQUATIONS AND MEASURE THEORY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any four** questions out of five questions. **Each** question carries **one** mark. **(4×1=4)**

1. Let $f_n(x) = \frac{e^{-nx}}{\sqrt{x}}$. Check the uniform convergence of this sequence of functions.
2. Find $\lim \sup\left(1+\frac{1}{n}\right)$.
3. Define σ-finite measure.
4. Write Fredholm integral equation.
5. Define characteristic function of an integral equation.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Define characteristic function. Give an example.
7. Define step function and integral of step function.
8. Let μ be a measure defined on a σ-algebra X. If E, F ∈ X and E ⊆ F, then show that $\mu(E) \leq \mu(F)$.
9. If μ is a measure on X and A is a fixed set in X, then show that the function λ, defined by E ∈ X, by $\lambda(E) = \mu(A \cap E)$ is a measure on X.

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10. Let (X, **X**, μ) be a measure space and let (E_n) is a sequence in **X**. Show that $\mu(\liminf E_n) \leq \liminf \mu(E_n)$.
11. If f ∈ L, λ is defined on X to R, by $\lambda(E) = \int_E f\, d\mu$. Then show that λ is a charge.
12. If f is measurable, g is integrable and $|f| \leq |g|$, then show that f is integrable and $\int |f|\, d\mu \leq \int |g|\, d\mu$.
13. Define Symmetric-Kernel.
14. Find the Wronskian of $u(\xi) = \xi$, $v(\xi) = \frac{1}{\xi}$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Define Riemann-integral and Lebesgue integral of a function.
16. Define σ-algebra and give an example. Define Borel algebra.
17. Show that the following statements are equivalent for a function f on X to R.
    a) For every α ∈ R, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$, belongs to X.
    b) For every α ∈ R, the set $A_\alpha = \{x \in X : f(x) \geq \alpha\}$, belongs to X.
    c) For every α ∈ R, the set $A_\alpha = \{x \in X : f(x) \leq \alpha\}$, belongs to X.
    d) For every α ∈ R, the set $A_\alpha = \{x \in X : f(x) < \alpha\}$, belongs to X.
18. Let μ be a measure defined on a σ-algebra X. If (E_n) is an increasing sequence in X, then show that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.
19. Define Charge. Show that sum and difference of two charge is charge.
20. Let λ denote Lebesgue measure defined on the Borel algebra B of R. If E consist of a single point then show that E ∈ B and λ(E) = 0.
21. Let (A_n) be a sequences of subsets of a set X. If A consist of all x ∈ X, which belongs to infinitely many of the sets A_n, show that $A = \bigcup_{m=1}^{\infty}\left[\bigcap_{n=m}^{\infty} A_n\right]$.
22. Show that a sum f + g of functions in L belongs to L and $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

23. If f is an **X**-measurable real valued functions and if f(x) = 0 for μ-almost all x in X, then show that f ∈ L (X, **X**, μ) and $\int f d\mu = 0$.
24. Form the Volterra integral equation corresponding to the initial value problem $\frac{d^2y}{dx^2} + \lambda y = 0$, with $y(0) = 0$, $y'(0) = 0$.
25. Form the differential equation with boundary condition, from the Fredholm integral equation $y(x) = \lambda\int_0^x \frac{\xi}{l}(l-x)\, y(\xi)\, d\xi + \lambda\int_x^l \frac{x}{l}(l-\xi)\, y(\xi)\, d\xi$.
26. Write down the four properties that have to be satisfied by Green's function of a second order differential equation with homogeneous boundary conditions.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Let f and g be measurable real-valued functions and let c be any real number. Then show that, the functions cf, f^2 and f + g are measurable.
28. If φ and ψ are simple functions in M^+ (X, **X**) and c ≥ 0, then show that
    a) $\int c\phi\, d\mu = c\int \phi\, d\mu$.
    b) $\int (\phi + \psi)\, d\mu = \int \phi\, d\mu + \int \psi\, d\mu$.
29. State and prove Lebesgue dominated convergence theorem.
30. Obtain the most general solution of $y(x) = \lambda\int_0^1 (1-3x\xi)\, y(\xi)\, d\xi + F(x)$, Where $F(x) = 0$ under the assumption $\lambda = 2, -2$. $\lambda \neq \pm 2$.