Reg. No. : ....

Name : ....

IV Semester B.Sc. Honours in Mathematics Degree (CBCSS – Supplementary/Improvement) Examination, April 2023
(2017 – 2020 Admissions)
BHM 401 : ADVANCED REAL ANALYSIS AND METRIC SPACES

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any** 4 questions out of 5 questions. **Each** question carries **1** mark : **(4×1=4)**

1. State Cauchy Schwarz inequality.
2. Define derived set.
3. Define norm of a partition.
4. Find $\lim \frac{x}{x+n}\ \forall x \in \mathbb{R}, x \geq 0$.
5. Define open set.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks : **(6×2=12)**

6. Evaluate $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$.
7. Prove that step functions are Riemann integrable.
8. Let S be any nonempty set and B(S) denote the set of all real – or complex – valued functions on S, each of which is bounded, define $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$, $f, g \in B(S)$. Show that d(f, g) is metric on B(S).
9. Check whether the sequence $(g_n) = x^n$ converge uniformly on [0, 1].
10. Prove that a convergent sequence in a metric space is a Cauchy sequence.

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11. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
12. Is the intersection of an infinite number of open sets open ? Justify your answer.
13. Prove that A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.
14. If $\sum a_n$ is an absolutely convergent series, then prove that the series $\sum a_n \sin nx$ is uniformly convergent.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks : **(8×4=32)**

15. Prove that if a function is Riemann integrable on [a, b] then prove that it is bounded on [a, b].
16. State and prove Squeeze theorem.
17. State and prove fundamental theorem of Calculus (first form).
18. State and prove Cauchy-Hadamard theorem.
19. Find $\lim \frac{\sin(nx+n)}{n}$ for $x \in \mathbb{R}$. Is this convergence uniform ? Justify.
20. Let X be a nonempty set and d denote the discrete metric. Then show that sequence $x_n \to x$ in (X, d) if and only if all the x_n, except possibly finitely many, are equal to x.
21. Let N denote the set of natural numbers. Define $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$, $m, n \in N$. Is (N,d) a complete metric space ? Justify.
22. State and prove Minkowski's Inequality (finite sum).
23. Let F be a subset of the metric space (X, d). Then prove that the set of limit points of F, is a closed subset of (X, d).
24. Give an example of a subset Y of a metric space (X, d) for which $\overline{Y^{o}} \neq (\overline{Y})^{o}$.

25. If a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then prove that the sequence converges to the same limit as the subsequence.
26. The space B(S) of all real-or complex-valued functions f on S, each of which is bounded, with the uniform metric $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$, $f, g \in B(S)$. Prove that B(S) with this metric is complete.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks : **(2×6=12)**

27. Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then prove that $f \in R[a, b]$ if and only if the restrictions to [a, c] and [c, b] are both Riemann integrable.
28. State and prove cauchy Criterion for uniform convergence of sequence of bounded functions.
29. Define pseudometric. Let X be the space of all sequences of numbers. Let $x = \{x_i\}$ and $y = \{y_i\}$ be elements of X. Define $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$. Is d a pseudometric on X ? Justify.
30. Let (X, d_x) and (Y, d_y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to f(a).