K23U 3610

Reg. No. : ....

Name : ....

III Semester B.Sc. Honours in Mathematics Degree (CBCSS – Supplementary) Examination, November 2023 (2018 – 2020 Admissions)

BHM 305 : ADVANCED LINEAR ALGEBRA

Time : 3 Hours

Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark.

1. Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is a linear operator on V defined by $T(A) = AB - BA$, and if f is the trace function, what is $T^t f$?

2. Let $T : R^2 \to R^2$ be defined by $T(x_1, x_2) = (x_1 + x_2, x_1 + x_2)$. Find characteristic values of T.

3. Let T be a linear operator on R^2 defined by $T(x_1, x_2) = (-x_2, x_1)$. Find the invariant subspaces of R^2.

4. Let $(|)$ be the standard inner product on R^2. Let $\alpha = (1, 2)$, $\beta = (-1, 1)$. If γ is a vector such that $(\alpha|\gamma) = -1$ and $(\beta|\gamma) = 3$, find γ.

5. Find a unitary matrix which is not orthogonal.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks.

6. Let V be a finite dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha)$, $f \in V^*$. Prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.

P.T.O.

7. If S is any subset of a finite dimensional vector space V, prove that $(S^\circ)^\circ$ is the subspace spanned by S.

8. Let T be a linear operator on a finite dimensional space V. If T is diagonalizable and if $c_1, \ldots, c_k$ are distinct characteristic values of T, prove that there exist linear operators $E_1, \ldots, E_k$ on V such that $T = c_1E_1 + \ldots + c_kE_k$.

9. Let V be a real or complex vector space with an inner product. Show that $||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2||\alpha||^2 + 2||\beta||^2$ for any $\alpha, \beta \in V$.

10. Prove that an orthogonal set of nonzero vectors is linearly independent.

11. Let V be a vector space and S be any nonempty subset of V. Prove that $S^\perp$ is a subspace of V.

12. Let V be a finite dimensional inner product space. If T and U are linear operators on V and c is a scalar, prove that :
    i) $(T + U)^* = T^* + U^*$
    ii) $(cT)^* = \overline{c}T^*$

13. Let T be a linear operator on C^2 defined by $T\epsilon_1 = (1 + i, 2)$, $T\epsilon_2 = (i, i)$. Using the standard inner product, find the matrix of T^* in the standard ordered basis.

14. Let U be a linear operator on an inner product space V. Prove that U is unitary if and only if the adjoint U^* of U exists and $UU^* = U^*U = I$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks.

15. Let $g, f_1, \ldots, f_r$ be linear functionals on a vector space V with respective null spaces $N_1, \ldots, N_r$. Prove that g is a linear combination of $f_1, \ldots, f_r$ if and only if N contains $N_1 \cap \ldots \cap N_r$.

16. Let V and W be finite dimensional vector spaces over the field F, and let T be a linear transformation from V into W. Prove that :
    i) rank (T^t) = rank(T)
    ii) the range of T^t is the annihilator of the null space of T.

17. Let T be a linear operator on R^3 which is represented in the standard basis by the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

18. Let T be a linear operator on the space V, and let $W_1, \ldots, W_k$ and $E_1, \ldots, E_k$ be on V such that
    i) $E_i^2 = E_i$ for each i;
    ii) $E_iE_j = 0$ if $i \neq j$;
    iii) $I = E_1 + \ldots E_k$;
    iv) the range of E_i is W_i.

    Prove that each subspace W_i is invariant under T if and only if $TE_i = E_iT$ for $i = 1, \ldots, k$.

19. Let T be a linear operator on the finite dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Prove that there is a diagonalizable operator D on V and a nilpotent operator N on V such that
    i) $T = D + N$
    ii) $DN = ND$.

20. Verify that standard inner product on F^n is an inner product.

21. State and prove polarization identities.

22. Let V be an inner product space, and let α, β be any vectors in V and c be any scalar. Then prove that
    i) $||c\alpha|| = |c|\,||\alpha||$
    ii) $||\alpha|| > 0$ for $\alpha \neq 0$
    iii) $|(\alpha|\beta)| \leq ||\alpha||\,||\beta||$.

23. Let W be a subspace of an inner product space V and β be a vector in V. Prove that the vector α in W is a best approximation in W if and only if $\beta - \alpha$ is orthogonal to every vector in W.

24. Let V be a finite dimensional inner product space and T be a linear operator on V. If T is invertible, show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

25. Let V be a finite dimensional inner product space. Prove that an idempotent operator E on V is self adjoint if and only if $EE^* = E^*E$.

26. Prove that T is normal if and only if $T = T_1 + iT_2$ where T_1 and T_2 are self adjoint operators which commute.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks.

27. Suppose V is a finite dimensional vector space over the field F. Prove that a linear operator T on V is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) \ldots (x - c_k)$, where $c_1, \ldots, c_k$ are distinct elements of F.

28. State and prove primary decomposition theorem.

29. State Gram-Schmidt orthogonalization process and using this process to the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$ to obtain an orthonormal basis for R^3 with standard inner product.

30. Let V be a complex inner product space and T be a self adjoint linear operator on V show that
    i) $||\alpha + iT\alpha|| = ||\alpha - iT\alpha||$ for every α in V.
    ii) $\alpha + iT\alpha = \beta + iT\beta$ if and only if $\alpha = \beta$.
    iii) $I + iT$ is nonsingular.