K22U 3680

Reg. No. : ....

Name : ....

Third Semester B.Sc. Honours in Mathematics Degree (C.B.C.S.S. – Supplementary/Improvement) Examination, November 2022

(2017-2020 Admissions)

BHM 305 : ADVANCED LINEAR ALGEBRA

Time : 3 Hours Max. Marks : 60

Answer **any 4.** **(4×1=4)**

1. Define a Linear Transformation.
2. Let F be a field and let f be the linear functional on F^2 be defined by $f(x_1, x_2) = ax_1 + bx_2$. Let T be defined by $T(x_1, x_2) = (-x_2, x_1)$ and let $g = T^t f$. Find $g(x_1, x_2)$.
3. Consider R^n as a vector space over R. What is the characteristic polynomial for the identity operator ?
4. Is a complex symmetric matrix self-adjoint ?
5. Let A be an n×n matrix. What is the normal of A ?

Answer **any 6** short answer questions out of 9 : **(6×2=12)**

6. If S is any subset of a finite dimensional vector space V, prove that $(S^\circ)^\circ$ is the subspace spanned by S.
7. Show that similar matrices have the same characteristic polynomial.
8. Let F be a field and V is a vector space over F. What is the inner product on V ?
9. Show that an orthogonal set of non-zero vector is linearly independent.
10. Let V be a finite dimensional inner product space and E the orthogonal projection of V on a subspace W. Show that $(E\alpha|\beta) = (\alpha|E\beta)$ for any vector α, β in V.
11. Let V be a finite dimensional inner product space. If T and U are linear operators on V. Show that $(T + U)^* = T^* + U^*$.

P.T.O.

12. Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the parallelogram law $||\alpha + \beta||^2 + ||\alpha - \beta||^2 = 2\,||\alpha||^2 + 2||\beta||^2$.
13. Let V be the vector space of continuous real valued functions on the real line. Let T be the linear operator on V defined by $(Tf)(x) = \int_0^x f(t)dt$. Prove that T has no characteristic values.
14. Show that every positive matrix is the square of positive matrix.

Answer **any 8** questions out of 12 : **(8×4=32)**

15. Let V be a finite dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha)$, f in V^*. Show that the mapping $\alpha \to L_\alpha$ is an isomorphism of V onto V^{**}.
16. Let V be a finite dimensional vector space over the field F. Let ß be an ordered basis for V with dual basis ß*, and let ß′ be an ordered basis for W with dual basis ß′*. Let T be the linear transformation from V into W; let A be the matrix of T relative to ß, ß′ and let B be the matrix of T^t relative to ß*, ß′*. Show that $B_{ij} = A_{ji}$.
17. Let A be the real 3×3 matrix $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$. Find the characteristic value of A.
18. State and prove Cauchy-Schwarz inequality.
19. Let T be the linear operator on R^4 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$. Under what condition on a, b and c is T diagonalizable ?
20. Let V be a finite dimensional inner product space, and T be a linear operator. Show that there exists a unique linear operator T^* on V such that $(T\alpha|\beta) = (\alpha|T^*\beta)$.
21. Let U be a linear operator on an inner product space V. Show that U is unitary if and only if the adjoint U^* of U exists and $UU^* = U^*U = 1$.

22. Let V be a inner product space and T a self-adjoint linear operator on V. Show that each characteristic value of T is real, and characteristic vectors of T associated with distinct characteristic value are orthogonal.
23. Let V be a finite dimensional inner product space, and T a linear operator on V, and ß an orthonormal basis for V. Suppose that the matrix A of T is the basis ß is the upper triangular. Show that T is normal if and only if A is a diagonal matrix.
24. Find the minimal polynomial for the matrix $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.
25. Let g, $f_1, \ldots, f_r$ be linear functional on a vector space V with respective null spaces N, $N_1, \ldots, N_r$. Prove that g is a linear combination of $f_1, \ldots, f_r$ if and only if N contains $N_1 \cap \ldots \cap N_r$.
26. Apply the Gram-Schmidt process to the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$, to obtain an orthonormal basis for R^3 with the standard inner product.

Answer **any 2** questions out of 4. **(2×6=12)**

27. Let T be a linear operator on R^3 which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is a characteristic vector of T.
28. State and prove Cayley Hamilton Theorem.
29. Let V be a finite dimensional vector space over the field F, and T be a linear operator on V. Prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomial over F.
30. Let V and W be a finite dimensional inner product spaces over the same field, having the same dimension. If T is a linear transformation from V into W. Prove that the following are equivalent.
    i) T preserves inner product.
    ii) T is an isomorphism.
    iii) T carries every orthonormal basis for V onto an orthonormal basis for W.
    iv) T carries some orthonormal basis for V onto an orthonormal basis for W.