Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (CBSS – Supple. (One Time Mercy Chance)/Imp.) Examination, April 2024
(2017 to 2022 Admissions)
MATHEMATICS
MAT2C07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Define Lebesgue outer measure. Show that $m^*(A) \leq m^*(B)$ if $A \subseteq B$.
2. Prove that, for any set A and any $\varepsilon > 0$ there is an open set O containing A and such that $m^*(O) \leq m^*(A) + \varepsilon$.
3. Show that if f is integrable, then f is finite valued a.e.
4. Show that here exist a smallest ring and a smallest σ-ring containing a given class of subsets of a space.
5. Define measure space and measurable space. Give examples.
6. Prove that if $\mu(x) < \infty$ and $0 < p < q \leq \infty$, then $L^q(\mu) \subseteq L^p(\mu)$.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Prove that the following statements regarding the set E are equivalent.
   i) E is measurable
   ii) $\forall \varepsilon > 0$, there exists O, an open set, $O \supseteq E$ such that $m^*(O - E) \leq \varepsilon$

   iii) there exists G, a G_δ-set, $G \supseteq E$ such that $m^*(G - E) = 0$
   iv) $\forall \varepsilon > 0$, there exists F, a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \varepsilon$
   v) there exists F, a F_σ-set, $F \subseteq E$ such that $m^*(E - F) = 0$

   b) Show that every countable set has measure zero.

8. a) Show that the class M of Lebesgue measurable sets is a σ-algebra.
   b) Show that there exists uncountable sets of zero measure.
9. a) Prove that Lebesgue outer measure is countably additive on disjoint measurable sets.
   b) Prove that not every measurable set is a Borel set.

Unit – II

10. a) Let f be bounded and measurable on a finite interval [a, b] and let $\varepsilon > 0$. Then show that there exist.
    i) a step function h such that $\int_a^b |f - h|\, dx < \varepsilon$,
    ii) a continuous function g such that g varnishes out side a finite interval and $\int_a^b |f - g|\, dx < \varepsilon$.

    b) Show that if $\alpha > 1$,
    $\int_0^1 \frac{\sin x}{1 + (nx)^\alpha}\, dx = O(n^{-1})$ as $n \to \infty$.
11. a) Show that $H(R) = [E : E \subseteq \bigcup_{n=1}^{n} E_n, E_n \in R]$.
    b) Let f be a bounded function defined on the finite interval [a, b], then prove that f is Riemann integrable over [a, b] if and only if it is continuous a.e.
12. a) Show that if μ is a σ-finite measure on R, then the extension $\overline{\mu}$ of μ is also σ-finite.
    b) If μ is a σ-finite measure on a ring R, then prove that it has a unique extension to the σ-ring S(R).

Unit – III

13. If $1 \leq p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$, then prove that there exist a function f and a subsequence $\{n_i\}$ such that $\lim f_{n_i} = f$a.e. Also prove that $f \in L^p(\mu)$ and $\lim \|f_n - f\|_p = 0$.
14. a) State and prove Holder's Inequality. When does the equality occur ?
    b) If $\rho(f, g)\ \|f - g\|_p$, then prove that, for $p \geq 1$, ρ is a metric on $L^p(\mu)$.
15. Let $p \geq 1$ and $f, g \in L^p(\mu)$, then prove that
    $$\left(\int |f + g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p\, d\mu\right)^{\frac{1}{p}}$$
    When does the equality occur ? Justify your answer.