Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Supple. (One Time Mercy Chance)/ Imp.)
Examination, October 2023
(2017 to 2022 Admissions)
MATHEMATICS
MAT1C01 : Basic Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Find all abelian groups, up to isomorphism of order 32.
2. Prove or disprove : Every abelian group of order 30 is cyclic.
3. Prove that the field Q is a field of quotients of Z.
4. Show that the group Z has no principal series.
5. Show that $\sqrt{2}$ is not a rational number.
6. Find all p such that x + 2 is a factor of $x^4 + x^3 + x^2 - x + 1$ in $Z_p[x]$.

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove the following : Let X be a G – set. Then $|G_x| = (G : G_x)$. If |G| is finite, then $|G_x|$ is a divisor of |G|.
   b) State and prove The first Sylow Theorem.

8. a) Let G be a group of order 108. Show that there exists a normal subgroup of order 27 or 9.
   b) Are the groups $Z_4 \times Z_{18} \times Z_{15}$ and $Z_3 \times Z_{36} \times Z_{10}$ isomorphic ? Why or why not ?
   c) Prove that the center of a finite non-trivial p-group G is non-trivial.
9. a) If H and K are finite subgroups of a group G, prove that $|HK| = \dfrac{(|H|)(|K|)}{(|H \cap K|)}$.
   b) Prove that every group of order 255 is cyclic.
   c) Show that every group of order 30 contains a subgroup of order 15.

Unit – II

10. a) Prove the following : Let F be a field of quotients of D and let L be any field containing D. Then there exists a map $\psi : F \to L$ that gives an isomorphism of F with a sub field of L such that $\psi(a) = a$ for all $a \in D$.
    b) Show that Q under addition is not a free abelian group.
    c) Let G = Z × Z × Z, H = Z × Z × {0} and N = {0} × Z × Z. Show that HN/N isomorphic to Z and H/(H∩N) isomorphic to Z.
11. a) Prove that any two fields of quotients of an integral domain D are isomorphic.
    b) Describe the field F of quotients of the integral subdomain $\{n + 2mi | n, m \in Z\}$ of C.
    c) State and prove the second Isomorphism Theorem.
12. a) Let $\phi : Z_{18} \to Z_{14}$ be a homomorphism where $\phi(1) = 8$.
       i) Find the kernel K of ϕ.
       ii) List the cosets in Z_{18}/K.
       iii) Find the group $\phi[Z_{18}]$.
    b) Show that S_n is not solvable for $n \geq 5$.
    c) Show that if G and G′ are free abelian groups, then G × G′ is free abelian.

Unit – III

13. a) Prove that the polynomial $\Phi_p(x) = \dfrac{x^p - 1}{x - 1}$ is irreducible over Q for any prime p.
    b) Prove the following : Let R be a commutative ring with unity. Then M is a maximal ideal of R if and only if R/M is a field.
14. a) Prove the following : Let $f(x) \in F[x]$, and let f(x) be of degree 2 or 3. Then f(x) is reducible over F if and only if it has a zero in F.
    b) If R is a ring with unity and N is an ideal of R containing a unit. Prove that N = R.
    c) Does $Z_5[x]/\langle x^3 + 3x + 2\rangle$ is a field ? Justify your answer.
    d) Describe all ring homomorphisms of Z × Z in to Z × Z.
15. a) Prove the following : If R is a ring with unity, then the map $\phi : Z \to R$ given by $\phi(n) = n.1$ for $n \in Z$ is a homomorphism of Z in to R.
    b) State and prove The Eisenstein Criterion.
    c) Show that $25x^5 - 9x^4 - 3x^2 - 12$ is irreducible over Q.

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