K23P 0203

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, April 2023
(2019 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Let X be a Banach space over K and let $A \in BL(X)$. Prove that $\sigma(A)$ is a compact subset of K.

2. If $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ in a normed space X and $k_n \to k$ in K, prove that $x_n + y_n \xrightarrow{w} x + y$ and $k_n x_n \xrightarrow{w} kx$.

3. Let X be a reflexive normed space. Prove that X is separable if and only if X′ is separable.

4. Let X and Y be a normed spaces and $F : X \to Y$ be linear. Prove that F is a compact map if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

5. Let H be a Hilbert space and $A \in BL(H)$. Prove that A is normal if and only if $\| A(x) \| = \| A^*(x) \|$ for all $x \in H$.

6. Let $A \in BL(H)$. If A is compact, prove that A^* is also compact.

P.T.O.

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. Let X be a nonzero Banach space over C and $A \in BL(X)$. Prove that
   a) $\sigma(A)$ is non empty.
   b) $r_\sigma(A) = \inf\limits_{n=1,2,\ldots} \|A^n\|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

8. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.
   b) Let X, Y and Z be normed spaces. Let $F \in BL(X, Y)$ and $G \in BL(Y, Z)$. Prove that
      i) $(GF)' = F'G'$
      ii) $\|F'\| = \|F\| = \|F''\|$ and
      iii) $F'' J_X = J_Y F$.

9. a) Let X be a normed space. If X′ is separable, prove that X is separable.
   b) Prove that $x_n \xrightarrow{w} x$ in l^1 if and only if $x_n \to x$ in l^1.

Unit – II

10. Let X be a normed space. Prove that X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence.

11. a) Let X be a uniformly convex normed space and (x_n) be a sequence in X such that $\|x_n\| \to 1$ and $\|x_n + x_m\| \to 2$ as $m, n \to \infty$. Prove that (x_n) is a Cauchy sequence.
    b) Let X and Y be normed spaces and $F \in BL(X, Y)$. If $F \in CL(X, Y)$, prove that $F' \in CL(X, Y)$. Also show that the converse holds if Y is a Banach space.

12. Let X be a normed space and $A \in CL(X)$. Prove that $\dim Z(A' - kI) = \dim Z(A - kI) < \infty$ for $0 \neq k \in K$.

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$.
    b) Let H be a Hilbert space and $A \in BL(H)$. Prove that $R(A) = H$ if and only if A^* is bounded below.

14. a) Let H be a Hilbert space and $A \in BL(H)$. Let A be self adjoint. Prove that $\|A\| = \sup \{|\langle A(x), x\rangle| : x \in H, \|x\| \leq 1\}$.
    b) State and prove generalized Schwarz inequality.

15. a) Let H be a Hilbert space and $A \in BL(H)$. Prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k : \overline{k} \in \sigma_e(A^*)\}$.
    b) Let $H \neq \{0\}$ and $A \in BL(H)$ be self adjoint. Prove that
       $\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A]$.