K23P 1413

Reg. No. : ...............................

Name : ...................................

**III Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.)**
**Examination, October 2023**
**(2020 Admission Onwards)**
**MATHEMATICS**
**MAT3E02 : Probability Theory**

Time : 3 Hours Max. Marks : 80

PART – A

I. Answer **any four** questions. **Each** question carries **four** marks.

a) Prove that the intersection of arbitrary number of fields is a field.

b) Prove that continuous real valued functions on $\mathbb{R}$ are Borel functions.

c) If X and Y are simple random variables then prove that $E(X \pm Y) = EX \pm EY$.

d) If a sequence of random variables $\{X_n\}$ converge to X in the $r^{th}$ mean, i.e., $X_n \xrightarrow{r} X$, then prove that $E|X_n|^r \longrightarrow E|X|^r$.

e) Suppose $\phi$ is the characteristic function of a general distribution function. F. Then show that $\phi$ is continuous.

f) Define weak convergence of a sequence of random variables. Prove that limit of a weak convergence sequence is unique.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

**Unit – I**

II. a) Given a class $\{A_i : i = 1, 2, ...., n\}$ of n sets then prove that there exists a class $\{B_i : i = 1, 2, ...., n\}$ of disjoint sets such that, $\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i$. Also prove that $\bigcup_{i=1}^{\infty} A_i = A_1 + A_1^c A_2 + A_1^c A_2^c A_3 + ....$

b) Prove that inverse mapping preserves all set relations.

P.T.O.



III. a) Let $\Omega$ be a sample space and $\{A_1, A_2, ...., A_n\}$ be a partition of $\Omega$. Show that any simple function X can be written in the form $X = \sum_{k=1}^{n} x_k I_{A_k}$, where $x_k$'s are distinct numerical constants.

b) Prove that any Borel function of a vector random variable (X, Y) is a random variable.

IV. a) Explain classical occupancy problem.

b) Prove that probability function defined on all intervals of the form (a, b), (a, b) $\in \mathbb{R}$, $a < b$ defines uniquely an extension to the minimal field containing all the intervals.

**Unit – II**

V. a) Show that the distribution function $F_X$ of a random variable X is non-decreasing, continuous on the right with $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. Also show that, every function F with above properties is the distribution function of a random variable on some probability space.

b) State and prove Jordan decomposition theorem.

VI. a) If two non-decreasing sequences of non-negative simple functions $\{X_n\}$ and $\{Y_n\}$ have the same limit X, then prove that $\lim EX_n = \lim EY_n = EX$.

b) Given that EX, EY and EX + EY exist, then show that $E(X + Y) = EX + EY$.

VII. a) Let $\{X_n\}$ be a sequence of random variable. Show that $X_n$ converges to 0 in probability $\left(X_n \xrightarrow{P} 0\right)$, if and only if $E\left(\frac{|X_n|}{1+|X_n|}\right) \to 0$, as $n \to \infty$.

b) State and prove monotone convergence theorem.



**Unit – III**

VIII. a) Explain bivariate characteristic function with an example.

b) Suppose $\phi$ is the characteristic function of a general distribution function F. Then prove the following.

i) $\phi$ is continuous

ii) $|\phi(u)| \le \phi(0) = F(+\infty) - F(-\infty)$
$\phi(-u) = \overline{\phi}(u)$, where $\overline{\phi}(u)$ is the complex conjugate of $\phi(u)$.

IX. a) State and prove Bochner's theorem.

b) Show that a sequence $\{F_n\}$ of distribution functions converges weakly if and only if it converges on a set D dense in $\mathbb{R}$.

X. a) State and prove Helly-Bray theorem.

b) Let $f_n(x)$ be non-negative measurable functions converging to f(x), for almost all x. If $\int_{\mathbb{R}} f_n(x)\,dx = c = \int_{\mathbb{R}} f(x)dx$, then $F_n(x) \xrightarrow{c} F(x)$, for all x, and also prove that $\sup_{B \in \mathcal{B}} \left|\int_B f_n(t)dt - \int_B f(t)dt\right| = \frac{1}{2}\int \left|f_n - f\right| dt \to 0$.

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