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K23P 0206

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.S.S.-Reg./Supple./Imp.) Examination, April 2023
(2019 Admission Onwards)
MATHEMATICS
MAT4E02 : Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. With the usual notations, prove that $\langle R_k z, R_j w\rangle = \langle z, R_{j-k} w\rangle = \langle R_{k-j} z, w\rangle$ for any integers j and k, z, $w \in l^2(Z_N)$.
2. Find D(U(z)) for z = (i, –1, –1, i).
3. Prove that the Trigonometric system is an orthonormal set in $L^2([-\pi, \pi))$.
4. Prove that every cauchy sequence $\{z_k\}_{k=M}^{\infty}$ converges in $l^2(Z)$.
5. Suppose f, $g \in L^1(R)$ and $\hat{f} = \hat{g}$ a.e. Prove that f = g a.e.
6. Suppose $f \in R$. Prove that every point of R is a Lebesgue point of f.

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove the following : Suppose $N = 2^n$, $1 \le p \le n$, and let $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ form a p^{th} stage wavelet filter sequence. Suppose $z \in l^2(Z_N)$. Then the output $\{x_1, x_2, \ldots, x_p, y_p\}$ of the analysis phase of the corresponding p^{th} stage wavelet filter can be computed using no more than $4N + \log_2 N$ complex multiplications.

   b) State and prove the necessary and sufficient condition for the existence of first stage wavelet basis for $l^2(Z_N)$.

P.T.O.

8. a) State and prove the folding lemma.

   b) Suppose M is a natural number and N = 2M and $z \in l^2(Z_N)$. Prove that $(z^*)^\wedge = \hat{z}(n+M)$ for all n.

   c) Given that $\hat{u} = (\sqrt{2}, 1, 0, 1)$ and $\hat{v} = (0, 1, \sqrt{2}, -1)$. Show that $\{v, R_2 v, u, R_2 u\}$ forms an ortho normal basis for $l^2(Z_N)$.

9. Construct Daubechies's D_6 wavelet basis on Z_N.

Unit – II

10. a) Define the spaces $L^1([-\pi, \pi))$ and $L^2([-\pi, \pi))$. Show that $L^2([-\pi, \pi)) \subseteq L^1([-\pi, \pi))$.

    b) Let $\{a_j\}_{j\in Z}$ be an ortho normal set in a Hilbert space H. Prove that the following are equivalent.

       i) $\{a_j\}_{j\in Z}$ is complete.

       ii) For all $f, g \in H, \langle f, g\rangle = \sum_{j\in Z} \langle f, a_j\rangle \overline{\langle g, a_j\rangle}$.

       iii) For all $f \in H, \|f\|^2 = \sum_{j\in Z} |\langle f, a_j\rangle|^2$.

    c) Suppose that $u, v \in l^2(Z)$. Prove that $B = \{R_{2k} v\}_{k\in Z} \cup \{R_{2k} u\}_{k\in Z}$ is a complete orthonormal set in $l^2(Z)$ if and only if the system matrix $A(\theta)$ is unitary for all θ.

11. a) With the usual notations prove that $V_{-l} \oplus W_{-l} = V_{-l+1}$ in $l^2(Z)$.

    b) Show that $L^2([-\pi, \pi))$ is a proper subset of $L^1([-\pi, \pi))$.

    c) Given that $z \in l^2(Z)$ and $w \in l^1(Z)$. Show that $z * w \in l^2(Z)$ and $\|z * w\| \le \|z\| \|w\|_1$.

12. a) Prove the following : Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta}\rangle = 0, \forall n \in Z$. Then $f(\theta) = 0$ a.e.

    b) Suppose $w \in l^1(Z)$.

       i) Prove that $\{R_k w\}, k \in Z$ is a complete orthonormal set for $l^2(Z)$ if and only if $|\hat{w}(\theta)| = 1$ for all $\theta \in [-\pi, \pi)$.

       ii) Prove that $\{R_{2k} w\}$, $k \in Z$ cannot be a complete orthonormal set in $l^2(Z)$.

Unit – III

13. a) Prove that $\wedge : L^2(R) \to L^2(R)$ is one-to-one and onto with inverse $\vee : L^2(R) \to L^2(R)$.

    b) State and Prove Fourier Inversion Theorem on $L^1(R)$.

14. a) Given that $G : R \to R$ by $G(x) = \frac{1}{\sqrt{2}} e^{\frac{-x^2}{2}}$. Prove that

       i) $\int_R G(x)dx = 1$

       ii) There exist $c_1 > 0$ such that $G(x) \le \frac{c_1}{(1+|x|)^2}$.

    b) Suppose $f \in L^1(R)$. Prove that $|\hat{f}(\zeta)| \le \|f\|$ for all ζ.

    c) Let $f(x) = \frac{1}{x}$ for x > 1. Show that $f \in L^2(R) \setminus L^1(R)$.

15. a) Suppose $f \in L^1(R)$ and $\hat{f} \in L^1(R)$. Prove that $\frac{1}{2\pi} \int_R \hat{f}(\xi) e^{ix\xi} d\xi = f(x)$ at every Lebesgue point x of f.

    b) Suppose $f \in L^1(R)$ and y, $\xi \in R$. Prove that $(\bar{f})^\wedge(\xi) = \overline{\hat{f}(\xi)}$ a.e.

    c) Suppose $f \in L^1(R)$. Prove that $\hat{f}$ is a continuous function on R.
