K22P 1603

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (C.B.S.S. – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks :

1. Prove that compact subset of metric spaces are closed.
2. Give an example of an open cover of the segment (0, 1) which has no finite subcover.
3. If $f(x) = |x^3|$. Show that $f^3(0)$ does not exist.
4. Using L'Hospital's rule, evaluate $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$.
5. Show that the polynomial $f(x) = x^5 + x^4 + x^3 + x + 1$ is of bounded variation on [0, 1].
6. If $\int_a^b f\, d\alpha = 0$ for every f which is monotonic on [a, b]. Prove that α must be a constant on [a, b].

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks :

Unit – I

7. a) Prove the following :
   i) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.
   ii) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
   iii) For any collection $G_1, G_2, \ldots, G_n$ of open sets, $\bigcap_1^n G_i$ is open.
   iv) For any collection $F_1, F_2, \ldots, F_n$ of closed sets, $\bigcup_1^n F_i$ is closed.

   b) Show that there exist a perfect set in R^1 which contain no segment.

P.T.O.

8. a) Let A be the set of all sequences whose elements are the digit 0 and 1. Prove that A is uncountable.
   b) Prove that countable union of countable set is countable.

9. a) Prove the following : Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = G \cap Y$ for some open subset G of X.
   b) Let A, B are two subsets of a metric space X. Prove that
   i) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$
   ii) $\overline{(A \cap B)} \subseteq \bar{A} \cap \bar{B}$.

Unit – II

10. a) State and prove the Generalized Mean Value Theorem.
    b) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}, a < x < b$.
    c) Prove the following : If f is monotonic on [a, b] and if α is continuous on [a, b], then $f \in R(\alpha)$.

11. a) If $f, g \in R(\alpha)$ on [a, b]. Prove that (i) $fg \in R(\alpha)$ (ii) $|f| \in R(\alpha)$ and $\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha$.
    b) Prove the following :
    i) If $f(x) \leq g(x)$ on [a, b], then $\int_a^b f\, d\alpha \leq \int_a^b g\, d\alpha$.
    ii) If $f \in R(\alpha)$ on [a, b] and if $a < c < b$, then $f \in R(\alpha)$ on [a, c] and [c, b].

12. a) State and prove L' Hospital's Rule.
    b) Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by $f(x) = x^a \sin(|x|^{-c})$, $x \neq 0$, and $f(0) = 0$. Prove the following :
    i) f is continuous if and only if $a > 0$.
    ii) $f'(0)$ exist if and only if $a > 1$.
    iii) f' is continuous if and only if $a > 1 + c$.
    iv) $f''(0)$ exist if and only if $a > 2 + c$.

Unit – III

13. a) State and prove the fundamental theorem of calculus.
    b) Let f be of bounded variation on [a, c] and [c, b]. Prove that $V_f(a, b) = V_f(a, c) + V_f(c, b)$.

14. a) Assume that f, g are each of bounded variation on [a, b]. Prove that $V_{f \pm g} \leq V_f \pm V_g$ and $V_{fg} \leq AV_f + BV_g$ for some $A, B \geq 0$.
    b) Show that the function $f(x) = x \sin\left(\frac{1}{x}\right)$, $x \neq 0$ and $f(0) = 0$ is not of bounded variation on $\left[0, \frac{2}{\pi}\right]$.

15. a) Given that $f = e^{2\pi it}$ if $t \in [0, 1]$ and $f = e^{2\pi it}$ if $t \in [0, 2]$. Prove that the length of g is twice as that of f.
    b) Prove that : Two paths f and g in R^n are equivalent if and only if they have the same graph.
    c) Examine whether the function $f(x) = x^2 \cos\left(\frac{1}{x}\right)$, $x \neq 0$, $f(0) = 0$ is of bounded variation on [0, 1].