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K22P 1601

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C01 : Basic Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each the elements.
2. Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Prove that $\sim$ is an equivalence relation on X.
3. Let N be a normal subgroup of G and H be any subgroup of G. Prove that $H \vee N = HN = NH$.
4. Let H^*, H and K be subgroups of G with H^* normal in H. Show that $H^* \cap K$ is normal in $H \cap K$.
5. Let $f(x) = 2x^2 + 3x + 4$, $g(x) = 3x^2 + 2x + 3$ in $\mathbb{Z}_6[x]$. Find $f(x) + g(x)$ and $f(x)g(x)$.
6. Let R be a ring with unity 1. Prove that the map $\phi : \mathbb{Z} \to R$ given by $\phi(n) = n \cdot 1$ for $n \in \mathbb{Z}$ is a homomorphism of $\mathbb{Z}$ into R.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $G_1, G_2, ..., G_n$ be groups. For $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in $\prod_{i=1}^{n} G_i$, define $(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)$ to be the element $(a_1b_1, a_2b_2, ..., a_nb_n)$. Prove that $\prod_{i=1}^{n} G_i$ is a group under this operation.
 b) State Fundamental theorem of finitely generated Abelian groups.
 c) Find all abelian groups of order 16 up to isomorphism.

P.T.O.

8. a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.
 b) Let X be a G-set and let $x \in G$. Prove that $|G_x| = (G : G_x)$. Also if $|G|$ is finite, show that $|G_x|$ is a divisor of $|G|$.
9. a) State and prove First Sylow theorem.
 b) Prove that no group of order 96 is simple.

Unit – II

10. Prove that any integral domain D can be embedded in a field F such that every element of F can be expressed as a quotient of two elements of D.
11. a) State and prove Third isomorphism theorem.
 b) Define free Abelian group. Prove that $\mathbb{Z}_n$ is not free Abelian.
 c) Let $G \neq \{0\}$ be a free abelian group with a finite basis. Prove that every basis of G is finite and all bases of G have the same number of elements.
12. State and prove Schreier theorem.

Unit – III

13. a) State and prove division algorithm for F[x].
 b) State Factor theorem and factorize $x^4 + 3x^3 + 2x + 4 \in \mathbb{Z}_5[x]$.
14. a) Let $f(x) \in F[x]$ and let f(x) be of degree 2 or 3. Prove that f(x) is reducible over F if and only if it has a zero in F.
 b) State and prove Eisenstein criterion for irreducibility.
 c) Prove that $25x^5 - 9x^4 - 3x^2 - 12$ is irreducible over $\mathbb{Q}$.
15. a) Let $R = \{a + b\sqrt{2} / a, b \in \mathbb{Z}\}$ and let R′ consists of all 2×2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{Z}$. Show that R is a subring of $\mathbb{R}$ and R′ is a subring of $M_2(\mathbb{Z})$. Also show that $\phi : R \to R'$, where $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism.
 b) Let R be a commutative ring with unity. Prove that M is a maximal ideal of R if and only if R/M is a field.