K23P 0204

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, April 2023
(2019 Admission Onwards)
MATHEMATICS
MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Define the Gradient Vector field. Find the gradient vector field of the function $f(x_1, x_2) = x_1 + 2x_2^2$, $x_1, x_2 \in R$.
2. Sketch the graph of the function $f : R^2 \to R$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$.
3. Define the term geodesic. Prove that geodesics have constant speed.
4. Compute $\nabla_v f$ where $f(x_1, x_2) = 2x_1^2 - 3x_1x_2^2$, $v = (1, 0, -1, 1)$.
5. Prove that $\beta(t) = (\sin t, -\cos t)$ is a reparametrization of $\alpha(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.
6. With usual notations, Prove that $d(f + g) = df + dg$.

PART – B

Answer **four** questions from this Part without omitting **any** Unit, **each** question carries **16** marks.

Unit – I

7. a) Find the integral curve through (1, 1) of the vector field $X(x_1, x_2) = (x_1, x_2, -x_2, -x_1)$.

   b) Let $a, b, c \in R$ such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the circle $x_1^2 + x_2^2 = 1$ are λ_1, λ_2 where λ_1, λ_2 are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

   c) State and Prove the Lagrange Multiplier Theorem.

P.T.O.

8. a) Prove the following : Let S be an n surface in R^{n+1}, $S = f^{-1}(c)$ where $f : U \to R$ is such that $\nabla_q \ne 0$ for all $q \in S$. Suppose $g : U \to R$ is a smooth function and $p \in S$ is an extreme point of g on S. Then there exist a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

   b) Sketch the cylinder $f^{-1}(0)$ where $f(x_1, x_2, x_3) = x_1 - x_2^2$.

   c) Find the orientations on the n-sphere $x_1^2 + x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = 1$.

9. a) Sketch the level curves (c = –1, 0, 1) and graph of the function $f(x_1, x_2) = x_1^2 + x_2^2$.

   b) i) Verify that a cylinder over an n – 1 surface in R^n is an n-surface in R^{n+1}.

   ii) Show that a surface of revolution is a 2-surface.

   c) Show that graph of any function $f : R^n \to R$ is a level set for some function $F : R^{n+1} \to R$.

Unit – II

10. a) Describe the spherical image of the 2-surface $f^{-1}(1)$, oriented by $\frac{-\nabla f}{\|\nabla f\|}$ where $f(x_1, x_2, x_3) = x_2^2 + x_3^2$.

    b) Let S denote the cylinder $x_1^2 + x_2^2 = 1$ in R^3. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$ for some $a, b, c, d \in R$.

11. a) Prove that in an n-plane parallel transport is path independent.

    b) Prove that The Weingarton map is self-adjoint.

12. a) Let $\alpha(t) = (x(t), y(t))$ be a local parametrization of the oriented plane curve C. Show that $\kappa \circ \alpha = x'y'' - x''y' / (x'^2 + y'^2)^{3/2}$.

    b) Show that

    i) $D_v(fX) = (\nabla_v f)X(p) + f(p)D_v X$

    ii) $\nabla_v(X.Y) = (D_v X).Y(p) + X(p).(D_v Y)$.

Unit – III

13. a) Prove the following : Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C. Then β is either one to one or periodic. Moreover, β is periodic if and only if C is compact.

    b) Find the Gaussian curvature of the ellipsoid $x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2 = 1$ oriented by its outward normal.

14. a) Let S be an oriented 2-surface in R^3 and let $p \in S$. Show that for each $v, w \in S_p$, $L_p(v) \times L_p(w) = K(p)\, v \times w$.

    b) Derive the formula for Gaussian curvature of an oriented n-surface in R^{n+1}.

15. a) Find the arc length of the curve $\alpha : [0, 1] \to R^2$ where $\alpha(t) = (t^2, t^3)$.

    b) Prove the following : Let S be an n surface in R^{n+1} and let $f : S \to R^k$. Then f is smooth if and only if $f \circ \phi : U \to R^k$ is smooth for each local parametrization $\phi : U \to S$.

    c) Compute $\int_\alpha (x_2 dx_1 + x_1 dx_2)$, where $\alpha(t) = (2\cos t, -\sin t)$, $0 \le t \le 2\pi$.

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