Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.)
Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT 3C13 : Complex Function Theory

Time : 3 Hours | Max. Marks : 80

PART – A

Attempt **any four** questions from this Part. **Each** question carries **4** marks.

1. Define the following terms :
   i) Period module of a meromorphic function
   ii) Discrete module.
2. Show that the series $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly and absolutely on a subset of the complex plane $\mathbb{C}$.
3. Is $\mathbb{C} - \{0\}$ is simply connected ? Justify your answer.
4. Is the sets $\{z : |z| < 1\}$ and $\mathbb{C}$ are homeomorphic ? Justify your answer.
5. Prove that a harmonic function u in $\mathbb{C}$ is infinitely differentiable.
6. Given that v_1 and v_2 are two harmonic conjugates of a harmonic function u. Prove that $v_2 - v_1 = c$, where c is a constant.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove the following :
   i) Let $S = \{z : \text{Re}z \geq a\}$ where $a > 1$. If $\varepsilon > 0$, then there is a number $\delta > 0$, $0 < \delta < 1$, such that for all $z \in S$, $\left|\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt\right| < \varepsilon$ whenever $\delta > \beta > \alpha$.
   ii) Let $S = \{z : \text{Re}z \leq A\}$ where $-\infty < A < \infty$. If $\varepsilon > 0$, then there is a number $k > 1$ such that for all $z \in S$, $\left|\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt\right| < \varepsilon$ whenever $\beta > \alpha > k$.

   b) Prove : A non-constant elliptic function has equally many poles as it has zeroes.

8. With the usual notations, prove that :
   a) $\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$
   b) $\wp'(z) = -\sigma(2z)/\sigma(z)^4$
   c) $\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0$
   d) $\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$

9. a) Prove that Riemann's zeta function ζ has no other zeroes outside the closed strip $\{z : 0 \leq z \leq 1\}$.
   b) Prove that if $\text{Re}\, z > 1$, then $\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-z}}\right)$ where p_n is a sequence of prime numbers.

Unit – II

10. State and prove Schwarz Reflection Principle.
11. a) Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Show that $\{(f_t', D_t) : 0 \leq t \leq 1\}$ is also a continuation along γ.
    b) Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Prove that there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D.
    c) Is the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$ is simply connected ? Justify your answer.
12. State and prove the Mittag-Leffler's theorem.

Unit – III

13. a) State and prove Jensen's formula.
    b) State and prove Maximum Principle (Second Version).
14. Prove that the Dirchlet problem can be solved in a unit disk.
15. a) Define the Poisson kernel $P_r(\theta)$. Prove that $P_r(\theta) = \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$.
    b) Prove that $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$.
    c) For $|z| < 1$ let $u(z) = \text{Im}\left[\left(\frac{1+z}{1-z}\right)^2\right]$. Show that u is harmonic.

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