



Reg. No. : .....

Name : .....

III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022  
(2019 Admission Onwards)

**MATHEMATICS**  
**MAT3C14 – Advanced Real Analysis**

Time : 3 Hours

Max. Marks : 80

**PART – A**

Answer any four questions from this Part. Each question carries 4 marks. (4×4=16)

- Let  $B$  be the uniform closure of an algebra  $A$  of bounded functions. Then prove that  $B$  is a uniformly closed algebra.
- Give an example of a functions with  $f_n$  converges to  $f$ , but  $f'_n$  does not converges to  $f'$ . Justify your answer.
- Define orthogonal system of functions. Give example with justification.
- Prove that  $\lim_{x \rightarrow +\infty} x^{-\alpha} \log x = 0$ .
- Prove that the existence of all partial derivatives does not imply the differentiability.
- Explain directional derivative of  $f$  at  $x$  in the direction of a unit vector  $u$  and continuously differentiable functions.

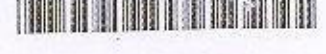
**PART – B**

Answer any four questions from this Part without omitting any Unit. Each question carries 16 marks. (4×16=64)

**Unit – I**

7. a) Suppose  $f_n \rightarrow f$  uniformly on a set  $E$  in a metric space. Let  $x$  be a limit point of  $E$ , and suppose that  $\lim_{t \rightarrow x} f_n(t) = A_n$ , ( $n = 1, 2, 3, \dots$ ). Then Prove that  $\{A_n\}$  converges and  $\lim_{t \rightarrow x} f(t) = \lim_{t \rightarrow \infty} A_n$ .

P.T.O.



- b) Suppose  $K$  is compact, and
- $\{f_n\}$  is a sequence of continuous functions on  $K$ ,
  - $\{f_n\}$  converges pointwise to a continuous function  $f$  on  $K$ ,
  - $f_n(x) \geq f_{n+1}(x)$  for all  $x \in K$ ,  $n = 1, 2, 3 \dots$ . Then prove that  $f_n \rightarrow f$  uniformly on  $K$ .
8. a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.  
b) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
9. Let  $A$  be an algebra of real continuous functions on a compact set  $K$ . If  $A$  separates points on  $K$  and if  $A$  vanishes at no point of  $K$ , then prove that the uniform closure  $B$  of  $A$  consists of all real continuous functions on  $K$ .

**Unit – II**

10. a) Suppose the series  $\sum_{n=0}^{\infty} c_n x^n$  converges for  $|x| < R$  and define  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ , ( $|x| < R$ ). Then prove that the series  $\sum_{n=0}^{\infty} c_n x^n$  converges uniformly on  $[-R + \epsilon, R - \epsilon]$ , no matter which  $\epsilon > 0$  is chosen. Also prove that the function  $f$  is continuous and differentiable in  $(-R, R)$  and  $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ ,  $|x| < R$ .
- b) Suppose the series  $\sum_{n=0}^{\infty} c_n x^n$  converges for  $|x| < R$  and define  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ , ( $|x| < R$ ). Then prove that  $f$  has derivatives of all orders in  $(-R, R)$  and derive the formulas.
11. State and prove Parseval's Theorem.
12. a) Define Gamma Function. Prove that  $\log \Gamma$  is convex on  $(0, \infty)$ .  
b) State and prove Stirling's Formula.

**Unit – III**

13. a) Let  $r$  be a positive integer. If a vector space  $X$  is spanned by a set of  $r$  vectors, then prove that  $\dim X \leq r$ .  
b) Suppose  $X$  is a vector space, and  $\dim X = n$ . Prove that  
i) A set  $E$  of  $n$  vectors in  $X$  spans  $X$  if and only if  $E$  is independent.



- $X$  has a basis and every basis consists of  $n$  vectors.
- If  $1 \leq r \leq n$  and  $\{y_1, y_2, \dots, y_r\}$  is an independent set in  $X$  then  $X$  has a basis containing  $\{y_1, y_2, \dots, y_r\}$ .

14. a) Suppose  $f$  maps an open set  $E \subset \mathbb{R}^n$  into  $\mathbb{R}^m$ . Then prove that  $f \in C(E)$  if and only if the partial derivatives  $D_j f_i$  exist and are continuous on  $E$  for  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ .  
b) Suppose  $f$  maps a convex open set  $E \subset \mathbb{R}^n$  into  $\mathbb{R}^m$ ,  $f$  is differentiable in  $E$  and there is a real number  $M$  such that  $\|f'(x)\| \leq M$  for every  $x \in E$ . Then prove that  $\|f(b) - f(a)\| \leq M\|b - a\|$  for all  $a \in E$ ,  $b \in E$ .
15. State and prove implicit function theorem.

