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Reg. No. : ....

Name : ....

K22P 3319

IV Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.)
Examination, April 2022
(2018 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer any four questions from this Part. Each question carries 4 marks.

1. Let X be a normed space and $A \in BL(X)$. Prove that A is invertible if and only if A is bounded below and surjective.

2. Let X and Y be normed spaces and F_1 and $F_2 \in BL(X, Y)$ and $k \in K$. Show that $(F_1 + F_2)' = F_1' + F_2'$, $(kF_1)' = kF_1'$.

3. Let X and Y be Banach spaces, $F : X \to Y$ is a compact map and R(F) is closed in Y. Prove that F is of finite rank.

4. If X is an infinite dimensional normed space and $A \in CL(X)$. Prove that $0 \in \sigma_a(A)$.

5. Let H be a Hilbert space. If each (A_n) is self adjoint operator in BL(H) and $||A_n - A|| \to 0$, then prove that A is self adjoint.

6. Prove that the adjoint of Hilbert Schmidt operator on a separable Hilbert space is Hilbert Schmidt operator. (4×4=16)

P.T.O.

PART – B

Answer any four questions from this Part without omitting any Unit. Each question carries 16 marks.

UNIT – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.
   b) Let X be a Banach space over K and $A \in BL(X)$. Show that $\sigma(A)$ is a compact subset of K.

8. a) Let X be a normed space and X′ is separable, prove that X is separable.
   b) Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of K^n with the norm $|| \ ||_p$ is linearly isomorphic to K^n with the norm $|| \ ||_q$.

9. a) Let X be a normed space and (x_n) be a sequence in X. Then prove that (x_n) is weak convergent in X if and only if
      i) (x_n) is a bounded sequence in X and
      ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x′ in some subset of X′ whose span is dense in X′.
   b) Let (x'_n) be a sequence in a normed space X. If
      i) (x'_n) is bounded and
      ii) $(x'_n(x))$ is a Cauchy sequence in K for each x in a subset of X whose span is dense in X.
      Then, prove that (x'_n) is weak* convergent in X′. Is the converse true ? Justify your answer.

UNIT – II

10. a) Let X be a reflexive normed space. Prove that every closed subspace of X is reflexive.
    b) Examine the reflexivity of $L^p([a, b])$, $1 \le p \le \infty$.

11. a) When a normed space X is said to be uniformly convex ?
    b) Let X be a Banach space which is uniformly convex in some equivalent norm. Then prove that X is reflexive. Is the converse true ? Justify your answer.

12. Let X be a normed space, Y be a Banach space and $F \in BL(X, Y)$, then prove that
    a) CL(X, Y) is a closed subspace of BL(X, Y).
    b) $F \in CL(X, Y)$ if and only if $F' \in CL(Y', X')$.

UNIT – III

13. Let H be a Hilbert space and $A \in BL(H)$. Then prove the following,
    a) A is injective if and only if $R(A^*)$ is dense in H.
    b) The closure of R(A) equals $Z(A^*)^{\perp}$.
    c) $R(A) = H$ if and only if A^* is bounded below.

14. Let H be a Hilbert space and $A \in BL(H)$.
    a) If A is normal, x_1 and x_2 are eigenvectors of A corresponding to distinct eigenvalues, then prove that $x_1 \perp x_2$.
    b) Prove that every spectral value of A is an approximate eigenvalue of A.
    c) Define the numerical range of A and show that it is bounded, but not closed.

15. Let A be compact operator on non-zero Hilbert space.
    a) Prove that non-zero approximate eigenvalue of A is an eigenvalue of A and the corresponding eigenspace is finite dimensional.
    b) If A is self adjoint, then prove that $||A||$ or $-||A||$ is an eigenvalue of A. (4×16=64)