K23U 2648

Reg. No. : ....

Name : ....

V Semester B.Sc. Honours in Mathematics Degree
(C.B.C.S.S. – O.B.E. – Regular) Examination, November 2023
(2021 Admission)
5B21 BMH : ADVANCED LINEAR ALGEBRA

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Define eigenvalue of a square matrix.
2. Find the characteristic polynomial of $A = \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix}$.
3. State spectral theorem for symmetric matrices.
4. Prove that any projection is idempotent.
5. Find the Hermitian conjugate of $A = \begin{bmatrix} i & 5-3i & 2+i \\ 3 & 1+2i & 4-9i \end{bmatrix}$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Find the eigenvalues of $A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$.
7. Prove that the determinant of an $n \times n$ matrix A is equal to the product of eigenvalues.
8. With the usual inner product on $\mathbb{R}^4$, prove that the vectors $x = (1, -1, 2, 0)^T$ and $y = (-1, 1, 1, 4)^T$ are orthogonal.

P.T.O.

9. State and prove generalised Pythagoras theorem.
10. Prove that transpose of an orthogonal matrix is orthogonal.
11. If U and W are subspaces of V, then prove $U + W = \{u + w / u \in U, w \in W\}$ is a subspace of V.
12. Let $V = \mathbb{R}^3$ be an inner product space and $S = \text{Lin}\{u, w\}$, where $u = (1, 2, -1)^T$ and $w = (1, 0, 1)^T$. Find $S^{\perp}$.
13. If A is an $n \times n$ matrix with real entries and if λ is a complex eigenvalue with corresponding eigenvector v, then prove that $\bar{\lambda}$ is also an eigenvalue of A with corresponding eigenvector $\bar{v}$.
14. Prove that $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^2$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Find the eigenvalues and the eigenvectors for the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.
16. Prove that an $n \times n$ matrix A is diagonalisable if and only if it has n linearly independent eigenvectors.
17. Diagonalise the matrix $A = \begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$.
18. State and prove Cauchy Schwarz inequality.
19. Suppose V is an inner product space and that vectors $v_1, v_2, \ldots, v_k \in V$ are pairwise orthogonal and none is the zero vector. Then prove that $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set of vectors.
20. Use the Gram Schmidt process to find an orthonormal basis for the subspace of R^4 spanned by the vectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

21. Suppose U and W are subspaces of a vector space. Prove that sum of U and W is direct if and only if every vector z in the sum can be written uniquely as $z = u + w$, where $u \in U$ and $w \in W$.
22. Suppose that A is an $m \times n$ real matrix. Prove that $R(A)^{\perp} = N(A^T)$.
23. Suppose, A is an $m \times n$ real matrix of rank n. Prove that the matrix $P = A(A^TA)^{-1}A^T$ represents the orthogonal projection of $\mathbb{R}^m$ on to the range R(A) of A.
24. Prove that an $n \times n$ matrix P is unitary if and only if the columns of P are an orthonormal basis of $\mathbb{C}^n$.
25. State and prove spectral decomposition theorem.
26. Suppose E_1, E_2, E_3 are three matrices such that $E_iE_j = \begin{cases} E_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$, for i = 1, 2, 3. Prove that $(\alpha_1E_1 + \alpha_2E_2 + \alpha_3E_3)^n = \alpha_1^nE_1 + \alpha_2^nE_2 + \alpha_3^nE_3$ for any positive integer n and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks. **(2×6=12)**

27. Prove that eigenvectors corresponding to different eigenvalues are linearly independent.
28. Orthogonally diagonalise the matrix $A = \begin{bmatrix} 7 & 0 & 9 \\ 0 & 2 & 0 \\ 9 & 0 & 7 \end{bmatrix}$.
29. Let V be a finite dimensional inner product space and S be any subspace of V. Prove that $V = S \oplus S^{\perp}$.
30. Find the spectral decomposition of the matrix $A = \begin{bmatrix} 2 & 1+i & 0 \\ 1-i & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Deduce the spectral decomposition of A^3 and use it to find the matrix A^3.
