



Reg. No. :

Name :

IV Semester M.Sc. Degree (C.B.S.S. – Reg./Suppl. (Including Mercy
Chance)/Imp.) Examination, April 2021
(2017 Admission Onwards)

MATHEMATICS
MAT 4C 15 : Operator Theory

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Let X be a normed space and A be a bounded operator on X . If $k \in \sigma_a(A)$, then prove that there is a sequence $\{x_n\}$ in X with $\|x_n\| = 1$ for every n such that $\|(A - kI)(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$.
2. Let X, Y and Z be normed spaces. If $F \in BL(X, Y)$ and $G \in BL(Y, Z)$ then prove that $(GF)' = F'G'$, where F' denotes the transpose of the operator F .
3. State true or false and justify. "Every weak convergence sequence in the dual of a normed space is weak* convergent."
4. State true or false and justify. "Every finite dimensional normed space is reflexive."
5. State true or false and justify. "Every continuous linear map on a normed space is compact."
6. Let A be unitary operator on a Hilbert space H . Then prove that $\|A\| = 1$.
7. Prove that the numerical range of a bounded operator on a Hilbert space is bounded.
8. Define Hilbert Schmidt operator on a separable Hilbert space and give an example.



PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

9. Let X be a Banach space and $A \in BL(X)$. Then prove that
 - a) A is invertible if and only if A is bounded below and the range of A is dense in X .
 - b) The set of all invertible operators is open in $BL(X)$.
10. Let X be a Banach space over \mathbb{C} and $A \in BL(X)$. Then prove that
 - a) $\sigma_e(A) \subseteq \sigma_a(A) \subseteq \sigma(A)$.
 - b) $\sigma(A)$ is non-empty.
11. Let X and Y be normed spaces and $F \in BL(X, Y)$.
 - a) If X' is separable, then prove that X is separable.
 - b) Prove that $\|F'\| = \|F\| = \|F''\|$ and $F''J_X = J_YF$, where J_X and J_Y are the canonical embedding of X and Y into X'' and Y'' respectively.
12. Let X be a normed space.
 - a) If X is finite dimensional, then prove that the weak convergence and norm convergence are the same.
 - b) If X is separable, then prove that every bounded sequence in X' has a weak* convergent subsequence.

Unit – II

13. Let X be a normed space.
 - a) If X is reflexive, then prove that every bounded sequence in X has a weak convergent subsequence.
 - b) If X is uniformly convex and $\{x_n\}$ is a bounded sequence in X such that $\|x_n\| \rightarrow 1$ and $\|x_n + x_m\| \rightarrow 2$ as $n, m \rightarrow \infty$, then prove that $\{x_n\}$ is a Cauchy sequence.



14. Let X be a normed space.
 - a) If X is reflexive, then prove that its dual X' is also reflexive.
 - b) If X is Banach and uniformly convex, then prove that X is reflexive.
 15. Let X and Y be normed spaces and $F : X \rightarrow Y$ be linear.
 - a) Prove that F is a compact map if and only if for every bounded sequence $\{x_n\}$ in X , $\{F(x_n)\}$ has a subsequence which converges in Y .
 - b) If F is compact, then prove that F' is also a compact map.
 16. Let X be a normed space and $A \in CL(X)$. Then prove that
 - a) The eigenspectrum and the spectrum of A are countable sets and zero is the only possible limit point of it.
 - b) $\sigma(A) = \sigma(A')$.
- Unit – III**
17. Let H be a Hilbert space and $A \in BL(H)$.
 - a) Prove that there is a unique $B \in BL(H)$ such that $\langle A(x), y \rangle = \langle x, B(y) \rangle$, for every $x, y \in H$. Can we drop the completeness of H ? Justify.
 - b) If $R(A) = H$, then prove that A^* is bounded below, where A^* is the adjoint of A .
 18. Let H be a Hilbert space over \mathbb{C} and $A \in BL(H)$.
 - a) If A is self-adjoint, then prove that $\|A\| = \sup \{|\langle A(x), x \rangle| : x \in H, \|x\| \leq 1\}$.
 - b) Prove that there are unique self adjoint operators B and C on H such that $A = B + iC$.
 19. Let H be a Hilbert space over \mathbb{C} and $A \in BL(H)$.
 - a) Prove that $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$.
 - b) If A is a self-adjoint operator, then prove that $A^2 \geq 0$ and $A \leq \|A\|I$, where I is the identity operator on H .
 20. Let H be a Hilbert space and $A \in BL(H)$.
 - a) If A is compact, then prove that A^* is also a compact operator.
 - b) If A is a Hilbert-Schmidt operator, then prove that A^* is also a Hilbert-Schmidt operator.