Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS-Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)
MATHEMATICS
MAT 4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

(Answer **any four** questions from this Part. **Each** question carries **4** marks.)

1. Give an example of an operator A such that $\sigma_e(A)$ is a proper subset of $\sigma_a(A)$.
2. $x_n \xrightarrow{w} x$ and $k_n \to k$ in K then show that $k_n x_n \xrightarrow{w} kx$ in X.
3. Show that finite dimensional and strictly convex spaces are uniformly convex.
4. Define Rayleigh quotient of an operator.
5. Let E be a measurable subset of $\mathbb{R}$ and $H = L^2(E)$. Fix z in $L^\infty(E)$ and define $A(x) = zx$, $x \in H$. Show that A is unitary if and only if $|z| = 1$.
6. Let H be denote the Hilbert space of all doubly infinite square summable scalar sequences $x = (x(j))$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$ For x in H, let $A(x)(j) = x(j-1)$ for all j. Then show that A is a unitary operator on H. **(4×4=16)**

PART – B

(Answer **any four** questions from this Part without omitting **any Unit**. **Each** question carries **16** marks.)

UNIT – I

7. a) Let X a Banach space over K and $A \in BL(X)$. Let $k \in K$ such that $|k|^p > \|A^p\|$ for positive integer p. Then prove that $k \notin \sigma(A)$ and $(A - kI)^{-1} = -\sum_{n=0}^{\infty} \frac{A^n}{k^{n+1}}$ and for every $k \in \sigma(A)$, $|k| \le \inf_{n=1, 2, \ldots} \|A^n\|^{\frac{1}{n}} \le \|A\|$.

   b) Define dual basis of a normed linear space and give an example.

P.T.O.

8. a) Let $1 \le p \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then prove that the dual of K^n with the norm $\|.\|_p$ is linearly isometric to K^n with the norm $\|.\|_q$.

b) Let X, Y and Z be normed spaces. Let F_1 and F_2 be in BL(X, Y) and $k \in K$. Then prove that $(F_1+F_2)'=F_1'+F_2'$ and $(kF_1)'=kF_1'$. Let $F \in BL(X, Y)$ and $G \in BL(Y, Z)$. Then prove also that $(GF)' = F'G'$.

9. a) Let X and Y be normed spaces and $F \in BL(X, Y)$. Then prove that $\|F'\| = \|F\| = \|F''\|$ and $F''J_X = J_YF$, where J_X and J_Y are the canonical embeddings of X and Y into X″ and Y″, respectively.

b) Let X be a normed space and $\{x_n\}$ be a sequence in X. Then prove that $\{x_n\}$ is weak convergent in X if and only if (i) (x_n) is a bounded sequence in X and (ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x′ in some subset of X′ whose span is dense in X′. In that case, also prove that for every subsequence (x_{nk}) of (x_n), x belongs to the closure of ($\{x_{n1}, x_{n2}$, cdots}) and $\|x\| \le \lim\inf_{n\to\infty}\|x_n\|$.

UNIT – II

10. a) Let X be a reflexive normed space. Then prove that every closed subspace of X is reflexive.

b) Let X and Y be normed spaces and $F : X \to Y$ be linear, compact map then prove that F(U) is a totally bounded subset of Y. Also prove that if Y is a Banach and F(U) is a totally bounded subset of Y, then F is a compact map.

c) Define reflexive spaces and give an example.

11. a) Let X and Y be normed spaces and $F : X \to Y$ be linear. Let $F \in CL(X, Y)$, where CL(X, Y) denotes the set of all compact linear maps from a normed spaces X to a normed space Y. If $x_n \xrightarrow{w} x$ in X, then prove that $F(x_n) \to F(x)$ in Y.

b) Let X be a normed space and $A \in CL(X)$. If X is finite dimensional, then prove that $0 \in \sigma_a(A)$.

12. a) Give an example of a linear space which is not uniformly convex.

b) Let X be a normed space and $A \in CL(X)$. Then prove that $\{k : k \in \sigma_e(A'), k \ne 0\} = \{k : k \in \sigma_e(A), k \ne 0\}$, where A′ is the transpose of A.

UNIT – III

13. a) Let H be a Hilbert space. Consider $A \in BL(H)$. Then prove that A is invertible if and only if A^* is invertible and in that case $(A^*)^{-1} = (A^{-1})^*$.

b) Let H be a Hilbert space. Consider $A \in BL(H)$. Then prove that the closure of R(A) equals $Z(A^*)^\perp$ and closure of R (A^*) equals $Z(A)^\perp$.

c) Let H be a Hilbert space. Consider $A \in BL(H)$. Then prove that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. In that case prove that $\|A^2\| = \|AA^*\| = \|A\|^2$.

14. a) Let H be a Hilbert space. Let A and B be unitary. Then prove that AB is unitary. Also, A + B is unitary if and only if it is surjective and Re $\langle A(x), B(x)\rangle = \frac{-1}{2}$ for every $x \in H$ with $\|x\| = 1$.

b) State and prove Generalized Schwarz inequality.

15. a) Let H be a non-zero Hilbert space and $A \in BL(H)$ be self adjoint. Then prove that $\{m_A, M_A\} \subset \sigma_a(A) \subset [m_A, M_A]$.

b) Let A be a compact operator on a Hilbert space $H \ne \{0\}$. Then show that every non-zero approximate eigenvalue of A is, in fact, an eigenvalue of A and the corresponding eigenspace is finite dimensional.

c) Define Hilbert-Schmidt Operator and give an example. **(4×16=64)**