Africt in meant by Levi-Civita parallalism? State any tour properties of Levi-Civita perallalism.

) Let S be an n-surface in X\*\*\*, fet p, q eS, Let c be a placewise emporing parametrized outwo from p to q. Prove that the parallel remaport P, a S, -- S, along it is a vector space isomorphism which preserves dot products.

b). Prove that logar puremetrizations of plans curves are unique up to

of Let x(t) = (x(t), y(t)), t = 1 be a local parametrication of an oriented plane mayor C. Show that x = x = (x, y'' - y', x'')/(x''' - y'''') http) denotes the

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Name : .....

IV Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)

MATHEMATICS

MAT4C16: Differential Geometry

Time: 3 Hours Max. Marks: 80

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Answer any four questions. Each question carries 4 marks :

- 1. Sketch the level sets and graph of the function  $f: \mathbb{R}^2 \to \mathbb{R}$  defined by  $f(x_1, x_2) = x_1 x_2$ .
- 2. Show that the set of all unit vectors at all points of  $\mathbb{R}^2$  forms a 3-surface in  $\mathbb{R}^4$ .
- 3. Show that if  $\alpha: I \to \mathbb{R}^{n+1}$  is a parametrized curve with constant speed then  $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$  for all t in I.
- If X and Y are smooth vector fields tangent to an n-surface S in R<sup>n+1</sup> along a parametrized curve α: I → S, verify that (X.Y)' = X'.Y + X.Y', where X' denote the covariant derivative of X.
- 5. Find the length of the parametrized curve  $\alpha:[0,2]\to\mathbb{R}^2$  defined by  $\alpha(t)=(t^2,\,t^3).$
- 6. Define the differential  $d\phi$  of a smooth map  $\phi: U \to \mathbb{R}^m$  where U is open in  $\mathbb{R}^n$ . Also with usual notations, show that the value of  $d\phi$  ( $\mathbb{V}$ ) does not depend on the choice of the parametrized curve. (4×4=16)

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## PART - B

Answer any four questions without omitting any Unit. Each question carries

16 marks:

## Unit - I

- a) Show that for a smooth vector field X on an open set U of R<sup>n+1</sup> there exists a maximal integral curve of X through each point p of U.
- b) Consider the vector field  $\mathbb{X}$   $(x_1, x_2) = (x_1, x_2, 1, 0)$  on  $\mathbb{R}^2$ . For  $t \in \mathbb{R}$  and  $p \in \mathbb{R}^2$ , let  $\phi_t(p) = \alpha_p(t)$  where  $\alpha_p$  is the maximal integral curve of  $\mathbb{X}$  through p.
  - i) Show that for, each t,  $\phi_t$  is a one to one transformation from  $\mathbb{R}^2$  onto itself.
  - ii) Show that  $\phi_0=$  identity,  $\phi_{t_1}+_{t_2}=\phi_{t_1}\circ\phi_{t_2}$  for all  $t_1,\,t_2\!\in\!\mathbb{R}$  and  $\phi_{-t}=\phi_t^{-1}$  for all  $t\!\in\!\mathbb{R}.$
- 8. a) Let U be an open set in ℝ<sup>n+1</sup> and let f: U → ℝ be smooth. Let p ∈ U be a regular point of f and let c = f (P). Prove that the set of all vectors tangent to f<sup>-1</sup>(c) at p is [∇ f(p)]<sup>⊥</sup>.
  - b) Define an n-surface in  $\mathbb{R}^{n+1}$ . Let  $f: U \to \mathbb{R}$  be smooth where U is open in  $\mathbb{R}^{n+1}$ . Show that graph (f) is an n-surface in  $\mathbb{R}^{n+1}$ .
- 9. a) State and prove Lagrange multiplier theorem.
  - b) Let S be an n-surface in R<sup>n+1</sup>, let X be a smooth tangent vector field on S and p ∈ S. Prove the existence of the maximal integral curve of X through p.
  - c) Find two orientations on the n-sphere  $x_1^2 + .... + x_{n+1}^2 = 1$ .

## Unit - II

- 10. a) Define spherical image of an oriented n-surface in  $\mathbb{R}^{n+1}$  and illustrate with an example.
  - b) Let S be an n-surface in R<sup>n+1</sup>, let p ∈ S and let V∈ S<sub>p</sub>.Prove the existence of the maximal geodesic in S passing through p with initial velocity V.



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- a) What is meant by Levi-Civita parallelism? State any four properties of Levi-Civita parallelism.
  - b) Let S be an n-surface in  $\mathbb{R}^{n+1}$ , let p, q  $\in$  S. Let  $\alpha$  be a piecewise smooth parametrized curve from p to q. Prove that the parallel transport  $P_{\alpha}: S_p \to S_q$  along  $\alpha$  is a vector space isomorphism which preserves dot products.
- 12. a) Prove that the Weingarten map is self-adjoint.
  - b) Prove that local parametrizations of plane curves are unique up to reparametrization.
  - c) Let  $\alpha$  (t) = (x(t), y(t)), t  $\in$  I be a local parametrization of an oriented plane curve C. Show that k  $\circ \alpha = (x' \ y'' y' \ x'')/(x'^2 + y'^2)^{3/2}$ , k(p) denotes the curvature of C at p  $\in$  C.

## Unit - III

- 13. a) Let C be a connected oriented plane curve and let β: I → C be a unit speed global parametrization of C. Prove that β is either one to one or periodic.
  - b) Let  $\eta$  be the 1-form on  $\mathbb{R}^2 \{0\}$  defined by  $\eta = \left(-x_2 / \left(x_1^2 + x_2^2\right)\right) dx_1 + \left(x_1 / \left(x_1^2 + x_2^2\right)\right) dx_2. \text{ Prove that for } \alpha : [a,b] \to \mathbb{R}^2 \{0\} \text{ any closed piecewise parametrized curve in } \mathbb{R}^2 \{0\}, \ \int\limits_{\alpha} \eta = 2\pi \, k \text{ for some integer } k.$
- 14. a) On each compact oriented n-surface S in R<sup>n+1</sup>, prove that there exists a point p such that the second fundamental form at p is definite.
  - b) Find the Gaussian curvature of the ellipsoid  $x_1^2 + (x_2^2/4) + (x_3^2/9) = 1$ , oriented by its outward normal.
- 15. a) Find the Gaussian curvature of the parametrized torus  $\Psi$  in  $\mathbb{R}^3$  defined by  $\Psi(\theta, \phi) = ((a + b \cos \phi) \cos \theta, (a + b \cos \phi) \sin \theta, b \sin \phi)$ .
  - b) Let S be an n-surface in  $\mathbb{R}^{n+1}$  and let  $f: S \to \mathbb{R}^k$ . If  $f \circ \phi$  is smooth for each local parametrization  $\phi: U \to S$ , then prove that f is smooth. (4×16=64)