Unit – III

13. Suppose $f \in L^1(\mathbb{R})$ and $\{g_t\}_{t>0}$ is an approximate identity. Then prove that for every Lebesgue point x of f, $\lim_{t\to 0^+} g_t * f(x) = f(x)$.

14. a) Define $G : \mathbb{R} \to \mathbb{R}$ by $G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then prove that

    i) $\int_{\mathbb{R}} G(x)dx = 1$

    ii) There exists $c_1 > 0$ such that $G(x) \le \frac{c_1}{(1+|x|)^2}$.

    iii) $\hat{G}(\xi) = e^{-\xi/2}$, or $\hat{G} = \sqrt{2\pi} G$.

    b) Suppose $f, g \in L^1(\mathbb{R})$ and $\hat{f}, \hat{g} \in L^1(\mathbb{R})$. Then prove that

    i) $f, g, \hat{f}, \hat{g} \in L^2(\mathbb{R})$

    ii) $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$.

15. a) Suppose $f, g \in L^2(\mathbb{R})$. Then prove that

    i) $\langle \check{f}, \check{g} \rangle = \frac{1}{2\pi} \langle f, g \rangle$

    ii) $\|\check{f}\| = \frac{1}{\sqrt{2\pi}} \|f\|$

    b) Suppose $g, h \in L^1(\mathbb{R})$, and either $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Then prove that

    i) $(f * g)^\wedge = \hat{f}\,\hat{g}$.

    ii) $f*(g*h) = (f*g)*h$. **(4×16=64)**

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (CBSS - Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)
MATHEMATICS
MAT 4E02 : Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *1) Notations are as in prescribed text book.*
*2) Answer **any four** questions from Part – **A**. Each question carries 4 marks.*
*3) Answer **any four** questions from Part – **B** without omitting **any** Unit. **Each** question carries 16 marks.*

PART – A

1. Let $z, w \in \ell^2(\mathbb{Z}_N)$. Prove that $\langle R_k z, R_j w \rangle = \langle z, R_{j-k} w \rangle = \langle R_{k-j} z, w \rangle$ for any $k \in \mathbb{Z}$.

2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a complex inner product space X, and let $x \in X$. Prove that $\{x_n\}_{n=1}^{\infty}$ converges to x in X if and only if $\|x_n - x\|$ converges to 0 as $n \to \infty$, as a sequence of numbers.

3. Define $f(\theta) = 1/\sqrt{|\theta|}$ for $\theta \ne 0$, and $f(0) = 0$. Show that $f \in L^1([-\pi, \pi))$ but $f \notin L^2([-\pi, \pi))$.

4. Suppose $z \in \ell^2(\mathbb{Z})$. Then prove that $(z^*)^\wedge(\theta) = \hat{z}(\theta + \pi)$.

5. Define addition on $L^2(\mathbb{R})$ by $(f + g)(x) = f(x) + g(x)$. Define multiplication of a function $f \in L^2(R)$ by a scalar $\alpha \in \mathbb{C}$ by $(\alpha f)(x) = \alpha f(x)$. With these operations, prove that $L^2(\mathbb{R})$ is a vector space.

6. If $f \in L^1(\mathbb{R})$ is continuous at x, then prove that x is a Lebesgue point of f. **(4×4=16)**

PART - B

Unit – I

7. a) Suppose $M \in \mathbb{N}$ and $N = 2M$, and $u \in l^2(\mathbb{Z}_N)$ is such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(\mathbb{Z}_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Then prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

b) Let $w \in l^2(\mathbb{Z}_N)$. Then show that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if $|\hat{w}(n)| = 1$ for all $n \in \mathbb{Z}_N$.

8. Suppose $N = 2^n$, $1 \le p \le n$, and $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ form a p^{th} stage wavelet filter sequence. Suppose $z \in l^2(\mathbb{Z}_N)$. Then prove that the output $\{x_1, x_2, x_3, \ldots, x_p, y_p\}$ of the analysis phase of the corresponding p^{th} stage wavelet filter bank can be computed using no more than $4N + N\log_2 N$ complex multiplications.

9. a) Suppose N is divisible by 2, and $u_1 \in l^2(\mathbb{Z}_N)$.

i) Define $u_2 \in l^2(\mathbb{Z}_{N/2})$ by $u_2(n) = u_1(n) + u_1\left(n+\frac{N}{2}\right)$. Then prove that for all m $\hat{u}_2(m) = \hat{u}_1(2m)$.

ii) Suppose N is divisible by 2^l. Define $u_l \in l^2(\mathbb{Z}_{N/2^{l-1}})$ by $u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1\left(n+\frac{kN}{2^{l-1}}\right)$. Then prove that $\hat{u}_l(m) = \hat{u}_1(2^{l-1}m)$.

b) Suppose N is divisible by 2^p. Suppose $u, v \in l^2(\mathbb{Z}_N)$ are such that the system matrix A(n) in $A(n) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}(n) & \hat{v}(n) \\ \hat{u}(n+M) & \hat{v}(n+M) \end{bmatrix}$ is unitary for all n. Let $u_1 = u$ and $v_1 = v$ and, for all $l = 2, 3, \ldots p$, define u_l by equation $u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1\left(n+\frac{kN}{2^{l-1}}\right)$ and v_l similarly with v_1 in place of u_1. Then show that $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ is a p^{th} stage wavelet filter sequence.

Unit – II

10. a) Let $\{a_j\}_{j\in\mathbb{Z}}$ be an orthonormal set in a Hilbert space H. Then prove that the following are equivalent :

i) $\{a_j\}_{j\in\mathbb{Z}}$ is complete.

ii) For all $f, g \in H$, $\langle f,g\rangle = \sum_{j\in\mathbb{Z}} \langle f,a_j\rangle \overline{\langle g,a_j\rangle}$

iii) For all $\in H, \|f\|^2 = \sum_{j\in\mathbb{Z}} |\langle f,a_j\rangle|^2$.

b) Let $\Sigma_{n\in\mathbb{Z}} w(n)$ be a series of complex numbers. Prove that $\Sigma_{n\in\mathbb{Z}} w(n)$ converges if and only if, for all $\epsilon > 0$, there exists an integer N such that $\left|\sum_{n=-m}^{-k} w(n) + \sum_{n=k}^{m} w(n)\right| < \epsilon$ for all $m \ge k > N$.

11. a) Suppose $f : [-\pi, \pi) \to \mathbb{C}$ is continuous and bounded, say $|f(\theta)| \le M$ for all θ. If $\langle f, e^{in\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta = 0$ for all $n \in \mathbb{Z}$. Then prove that $f(\theta) = 0$ for all $\theta \in [-\pi, \pi)$.

b) Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$, such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.

12. a) Suppose $w, z \in l^1(\mathbb{Z})$ then prove that the set $\{R_{2k}w\}_{k\in\mathbb{Z}}$ is orthonormal if and only if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta+\pi)|^2 = 2$ for all $\theta \in [0\ \pi)$.

b) Suppose $u \in l^1(\mathbb{Z})$ and $\{R_{2k}u\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Define a sequence $v \in l^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ then prove that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a complete orthonormal system in $l^2(\mathbb{Z})$.