



Reg. No. :

Name :

**IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory**



Time : 3 Hours

Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Let X be a normed space over K . If $A, B \in BL(X)$ and $k \neq 0$, then prove that $k \in \sigma(AB)$ if and only if $k \in \sigma(BA)$.
2. $x_n \rightarrow^w x$ and $y_n \rightarrow^w y$ in a normed space X then show that $x_n + y_n \rightarrow^w x + y$.
3. Interpret uniform convexity geometrically.
4. Define numerical range of an operator on a Hilbert space and prove or disprove that it is closed subset of K .
5. Let E be a measurable subset of \mathbb{R} and $H = L^2(E)$. Fix z in $L^\infty(E)$ and define $A(x) = zx, x \in H$. Show that A is normal.
6. Let u_1, u_2, \dots constitute an orthonormal basis for H . Suppose that $A \in BL(H)$ is defined by a matrix M with respect to u_1, u_2, \dots . Assume that M is triangular. Then show that A is normal if and only if M is diagonal. **(4x4=16)**

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Then show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.
- b) Let X a Banach space. If $A, B \in BL(X)$, A is invertible and $\epsilon = \|(A - B)A^{-1}\| < 1$, then show that B is invertible, $B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n$, $\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \epsilon}$ and $\|B^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\| \epsilon}{1 - \epsilon}$.



8. a) State and prove Spectral radius formula.
 b) Let X be a normed space. Then prove that if X' is separable, so is X .
9. a) Show that the dual of c_0 with the norm $\|\cdot\|_\infty$ is linearly isometric to l^1 .
 b) Let X be a normed space and $\{x_n\}$ be a sequence in X . Then prove that $\{x_n\}$ is weak convergent in X if and only if
 i) $\{x_n\}$ is a bounded sequence in X and
 ii) there is some $x \in X$ such that $x'(x_n) \rightarrow x'(x)$ for every x' in some subset of X' whose span is dense in X' .

Unit - II

10. a) Let X be a Banach space which is uniformly convex in some equivalent norm. Then prove that X is reflexive.
 b) Define compact linear map and give an example.
11. a) Let X and Y be normed spaces and $F : X \rightarrow Y$. If $F \in CL(X, Y)$, then prove that $F' \in CL(Y', X')$. Also prove the converse if Y is a Banach space.
 b) Let X be normed space and $A \in CL(X)$, and $0 \neq k \in K$. If $\{x_n\}$ is a bounded sequence in X such that $A(x_n) - kx_n \rightarrow y$ in X , then prove that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow x$ in X and $A(x) - kx = y$.
12. a) Let X be a linear space, $A : X \rightarrow X$ linear and $A(x_n) = k_n x_n$ for some $0 \neq x_n \in X$ and $k_n \in K$, $n = 1, 2, \dots$. Let $k_n \neq k_m$ whenever $n \neq m$. Then prove that $\{x_1, x_2, \dots\}$ is linearly independent subset of X .
 b) Let X be a normed space and $A \in CL(X)$. Then prove that every eigenspace of A corresponding to a nonzero eigenvalue of A is finite dimensional.

Unit - III

13. a) Define invertible operator. Also give an example of an invertible operator.
 b) Let H be a Hilbert space. Consider $A \in BL(H)$. Then prove that $Z(A) = R(A^*)^\perp$ and $Z(A^*) = R(A)^\perp$. Also prove that A is injective if and only if $R(A^*)$ is dense in H , and A^* is injective if and only if $R(A)$ is dense in H .
 c) Define self-adjoint operator and give an example.



14. a) Let H be a Hilbert space. Consider $A \in BL(H)$ and A be self adjoint. Then prove that $\|A\| = \sup \{ |\langle A(x), x \rangle| : x \in H, \|x\| \leq 1 \}$.
 b) Let H be a Hilbert space and $\{A_n\}$ be a sequence in $BL(H)$ and $A \in BL(H)$ be such that $\|A_n - A\| \rightarrow 0$ as $n \rightarrow \infty$. If each A_n is self adjoint unitary or normal, then prove that A is self adjoint, unitary or normal respectively.
15. a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k : \bar{k} \in \sigma_e(A^*)\}$.
 b) Let H be a finite dimensional Hilbert space over K and $A \in BL(H)$. Suppose that there is an orthonormal basis for H consisting of eigen values of A . Then prove that A is a normal operator. If $K = \mathbb{R}$, then prove that A is in fact a self adjoint operator. (4×16=64)