Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Define a vector field and illustrate it with an example.
2. Let $f : U \to \mathbb{R}$ be a smooth function on U, U open in $\mathbb{R}^n$. Show that the graph of f is an n-surface in $\mathbb{R}^{n+1}$.
3. Show that the spherical image of an n-surface S with orientation $\mathbb{N}$ is the reflection through the origin of the spherical image of S with orientation $-\mathbb{N}$.
4. Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, t)$.
5. Define length of a parametrized curve in $\mathbb{R}^{n+1}$ and show that it is invariant under reparametrization.
6. Describe a parametrized torus in $\mathbb{R}^4$. **(4×4=16)**

PART – B

Answer **any four** questions without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Prove the existence of the maximal integral curve of $\mathbb{X}$ through p.

   b) Sketch typical level curves and the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = -x_1^2 + x_2^2$.

P.T.O.

8. a) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

b) Let $f : U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f.

i) Show that $\frac{d}{dt}(f \circ \alpha)(t) = \| \nabla f(\alpha(t)) \|^2$ for all $t \in I$.

ii) Show that for each $t_0 \in I$, the function f is increasing faster along α at $\alpha(t_0)$ then along any other curve passing through $\alpha(t_0)$ with the same speed.

9. a) State and prove the Lagrange multiplier theorem.

b) Prove that each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

c) Define an oriented n-surface. Give an example of an "unoriented 2-surface" with justification.

Unit – II

10. a) Prove that for a compact connected oriented n-surface S in $\mathbb{R}^{n+1}$ with $S = f^{-1}(c)$, $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function with $\nabla f(p) \neq 0$ for all $p \in S$, the Gauss map $N : S \to S^n$ is onto.

b) Prove that geodesics have constant speed.

11. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$ and let $v \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field $\mathbb{V}$ tangent to S along α, which is parallel and has $\mathbb{V}(t_0) = v$.

b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve and let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields tangent to S along α. Verify that

i) $(\mathbb{X} + \mathbb{Y})' = \mathbb{X}' + \mathbb{Y}'$ and

ii) $(f\,\mathbb{X})' = f'\,\mathbb{X} + f\,\mathbb{X}'$

for all smooth function f along α.

12. a) Prove that the Weingarten map is self-adjoint.

b) Define a local parametrization of plane curve. Find a global parametrization of the curve oriented by $\nabla f / \| \nabla f \|$ where f is the function defined by the left side of the equation $ax_1 + bx_2 = c$, $(a, b) \neq (0, 0)$.

Unit – III

13. a) On each compact oriented n-surface S in $\mathbb{R}^{n+1}$, prove that there exists a point p such that the second fundamental form at p is definite.

b) Define a differential 1-form. Prove that for each 1-form W on U(U open in $\mathbb{R}^{n+1}$) there exist unique functions $f_i : U \to \mathbb{R}$, $i = 1, 2, ..., n + 1$ such that $W = \sum_{i=1}^{n+1} f_i \, dx_i$.

14. a) Find the Gaussian curvature of the ellipsoid $(x_1^2 / a^2) + (x_2^2 / b^2) + (x_3^2 / c^2) = 1$ (a, b, c all $\neq 0$) oriented by its outward normal.

b) Let ψ be the parametrized torus in $\mathbb{R}^3$:

$\psi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$

Find its Gaussian curvature.

15. a) Define an n-surface S in $\mathbb{R}^{n+k}$ ($k \geq 1$). With usual notations express S in the form $S = \bigcap_{i=1}^{k} f_i^{-1}(c_i)$. Define the tangent space S_p at $p \in S$ and the normal space to S at p. Illustrate a 1-surface in $\mathbb{R}^3$ with its tangent space and normal space at a point p.

b) State and prove the inverse function theorem for n-surfaces. **(4×16=64)**
