ii) Suppose f, g $\in L^2(\mathbb{R})$. Then prove that inequality $\int_{\mathbb{R}} |f(x-y)g(y)|\,dy < \infty$ holds for a.e. $x \in \mathbb{R}$. In this case $f * g \in L^1(\mathbb{R})$ with $\| f*g \|_1 \leq \| f \|_1 \| g \|_1$.

iii) Suppose $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then prove that $\int_{\mathbb{R}} |f(x-y)g(y)|\,dy < \infty$ holds for a.e. $x \in \mathbb{R}$. In this case prove that, $f * g \in L^2(\mathbb{R})$ with $\| f*g \| \leq \| f \| \| g \|_1$.

b) Suppose f, g $\in L^2(\mathbb{R})$ and x, y $\in \mathbb{R}$. Then prove that

i) $\langle R_x f, R_y g \rangle = \langle f, R_{y-x}\, g \rangle$

ii) $\langle f, R_y g \rangle = f * \tilde{g}(y)$.

14. a) Suppose $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Then prove that $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi}\,d\xi = f(x)$ at every Lebesgue point x of f.

b) Suppose $f \in L^2(\mathbb{R})$ and let $\epsilon > 0$. Then prove that there exists a C^2 function g with compact support, satisfying $\|f - g\| < \epsilon$.

15. a) Suppose f, g $\in L^2(\mathbb{R})$. Then prove that

i) $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$

ii) $\|\hat{f}\| = \sqrt{2\pi}\, \|f\|$

b) Suppose $f \in L^2(\mathbb{R})$. Then prove that $f = (\hat{f})^{\vee}$ and $f = (\check{f})^{\wedge}$. **(4×16=64)**

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
Paper – MAT 4E 02 : Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 80

Instructions : 1) *Notations are as in prescribed text book.*
2) *Answer **any four** questions from Part – **A**. Each question carries **4** marks.*
3) *Answer **any four** questions from Part – **B** without omitting **any** Unit. Each question carries **16** marks.*

PART – A

1. Suppose z, w $\in \ell^2(\mathbb{Z}_N)$. For any $k \in \mathbb{Z}$, then prove that $z * \tilde{w}(k) = \langle z, R_K w \rangle$, where $R_K w(n) = w(n-k)$.

2. Let $\sum_{n \in \mathbb{Z}} w(n)$ be a series of complex numbers. Prove that $\sum_{n \in \mathbb{Z}} w(n)$ converges if and only if, for all $\epsilon > 0$, there exists an integer N such that $\left| \sum_{n=-m}^{-k} w(n) + \sum_{n=k}^{m} w(n) \right| < \epsilon$ for all $m \geq k > N$.

3. Suppose $M \in \mathbb{Z}, \{x_n\}_{n=M}^{\infty}$ is a sequence in a complex inner product space X, and $\{x_n\}_{n=M}^{\infty}$ converges in X to some $x \in X$. Prove that $\{x_n\}_{n=M}^{\infty}$ is a Cauchy sequence.

4. Suppose z, w $\in \ell^2(\mathbb{Z})$. Then prove that $\bar{z}, z^* \in \ell^2(\mathbb{Z})$ and $R_K Z \in \ell^2(\mathbb{Z})$, for all $k \in \mathbb{Z}$.

5. Prove that $\ell^2(\mathbb{Z})$ is a vector space with the usual component wise addition and scalar multiplication.

6. Write a short note on the p^{th} stage wavelet system for $\ell^2(\mathbb{Z})$. **(4×4=16)**

P.T.O.

PART – B

Unit – I

7. a) Suppose $M \in \mathbb{N}$, $N = 2M$ and $w \in l^2(\mathbb{Z}_N)$. Then show that $\{R_{2K}w\}_{K=0}^{M-1}$ is an orthonormal set with M elements if and only if $|\hat{w}(n)|^2 + |\hat{w}(n+M)|^2 = 2$ for $n = 0, 1, \ldots, M-1$.

   b) Define first stage wavelet basis for $l^2(\mathbb{Z}_N)$.

8. Explain the construction of Daubechies's D_6 wavelets on $\mathbb{Z}_N$.

9. Suppose N is divisible by 2^l, $g_{l-1} \in l^2(\mathbb{Z}_N)$ and the set $\{R_{2^{l-1}k}\, g_{l-1}\}_{k=0}^{\left(\frac{N}{2^{l-1}}\right)-1}$ is orthonormal with $N/2^{l-1}$ elements. Suppose $u_l v_l \in l^2\left(\mathbb{Z}_{N/2^{l-1}}\right)$ and the system matrix $A_l(n)$ in equation $A_l(n) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_l(n) & \hat{v}_l(n) \\ \hat{u}_l\left(n+\frac{N}{2^l}\right) & \hat{v}_l\left(n+\frac{N}{2^l}\right) \end{bmatrix}$ is unitary for all $n = 0, 1, \ldots, (N/2^l) - 1$. Define $f_l = g_{l-1} * U^{l-1}(v_l)$ and $g_l = g_{l-1} * U^{l-1}(u_l)$. Then prove that $\{R_{2^l k} f_l\}_{k=0}^{(N/2^l)-1} \cup \{R_{2^l k} g_l\}_{k=0}^{(N/2^l)-1}$ is an orthonormal set with $N/2^{l-1}$ elements.

Unit – II

10. a) Suppose H is a Hilbert space, $\{a_j\}_{j\in\mathbb{Z}}$ is an orthonormal set in H, and $z = (z(j))_{j\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Then prove that the series $\sum_{j\in\mathbb{Z}} z(j)a_j$ converges in H, and $\left\| \sum_{j\in\mathbb{Z}} z(j)a_j \right\|^2 = \sum_{j\in\mathbb{Z}} |z(j)|^2$.

    b) Suppose $z = (z(n))_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Then prove that the series $\sum_{n\in\mathbb{Z}} z(n)e^{in\theta}$ converges to an element of $L^2([-\pi, \pi))$.

11. a) Suppose H is a Hilbert space and $T : H \to H$ is a bounded linear transformation. Suppose the series $\sum_{n\in\mathbb{Z}} x_n$ converges in H. Then prove that $T\left(\sum_{n\in\mathbb{Z}} x_n\right) = \sum_{n\in\mathbb{Z}} T(x_n)$, where the series on the right converges in H.

    b) Show that the Fourier transform on $l^2(\mathbb{Z})$ is one to one and onto, with inverse. For $z \in l^2(\mathbb{Z})$, $z(n) = (\hat{z})^{\vee}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{z}(\theta)e^{-in\theta}d\theta$.

12. a) Suppose $M \in \mathbb{N}$ and $N = 2M$. Suppose $u, v \in l^1(\mathbb{Z})$ are such that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first stage wavelet system for $l^2(\mathbb{Z})$. Define $u_{(N)}, v_{(N)} \in l^2(\mathbb{Z}_N)$ by $u_{(N)}(n) = \sum_{k\in\mathbb{Z}} u(n+kN)$ and $v_{(N)}(n) = \sum_{k\in\mathbb{Z}} v(n+kN)$. Then prove that $\{R_{2K}v_{(N)}\}_{K=0}^{M-1} \cup \{R_{2K}u_{(N)}\}_{K=0}^{M-1}$ is a first stage wavelet basis for $l^2(\mathbb{Z}_N)$.

    b) Suppose $u_l, v_l \in l^1(\mathbb{Z})$ for each $l \in \mathbb{N}$, and the system matrix $A_l(\theta)$ defined in equation $A_l(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_l(\theta) & v_l(\theta) \\ \hat{u}_l(\theta+\pi) & \hat{v}_l(\theta+\pi) \end{bmatrix}$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = u_1$, $g_1 = v_1$ and inductively, for $l \in \mathbb{N}$, $l \geq 2$, define f_l and g_l by equation $f_l = g_{l-1} * U^{l-1}(v_1)$, $g_l = g_{l-1} * U^{l-1}(u_l)$. For each $l \in \mathbb{N}$, define V_{-l} as in equation $V_{-l} = \left\{E_{k\in\mathbb{Z}}\, z(k)R_{2^l k}\, g_l : z = (z(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})\right\}$. Suppose $\cap_{l\in\mathbb{N}} V_{-l} = \{0\}$. Define B as in equation $B = \{R_{2^l k} f_l : k \in \mathbb{Z}, l \in \mathbb{N}\}$. Then prove that B is a complete orthonormal set in $l^2(\mathbb{Z})$.

Unit – III

13. a) i) Suppose $f, g \in L^2(\mathbb{R})$. Then prove that inequality $\int_{\mathbb{R}} |f(x-y)g(y)|\,dy < \infty$ holds for all $x \in \mathbb{R}$. In this case $f * g$ is bounded and $|f * g(x)| \leq \|f\|\|g\|$ for all $x \in \mathbb{R}$.