K18P 0325

Reg. No. : ....

Name : ....

BRENNEN COLLEGE THALASSERY

Fourth Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, March 2018
MATHEMATICS
(2014 Admission Onwards)
MAT4E05 : Fourier and Wavelet Analysis

Time : 3 Hours Max. Marks : 60

***Instruction** : Notations are as in **prescribed** text book.*

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. Define the down sampling operator D and the up sampling operator U. Also find U∘D (z) for $z \in l^2(\mathbb{Z}_N)$.
2. For $u, v \in l^2(\mathbb{Z}_N)$, prove that $\langle \tilde{u}, R_{2k}\tilde{v} \rangle = \langle v, R_{2k}u \rangle$.
3. Suppose $\{z_k\}_{k=0}^{\infty}$ is a sequence in $l^2(\mathbb{Z})$ and suppose $z_k \to z$ in $l^2(\mathbb{Z})$. Prove that $\lim_{k\to\infty} z_k(n) = z(n)$ for each $n \in \mathbb{Z}$.
4. Show that $l^2(\mathbb{Z})$ is not closed with respect to convolution.
5. If $f \in L^1(\mathbb{R})$, show that $\left| \int_{\mathbb{R}} f(x)dx \right| \leq \int_{\mathbb{R}} |f(x)|dx$.
6. Suppose $\mu : \mathbb{R} \to \mathbb{C}$ is multiplicative, μ is not identically zero and μ is differentiable at zero. Prove that $\mu(x) = e^{cx}$ for some $c \in \mathbb{C}$. **(4×3=12)**

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **12** marks.

Unit – I

7. a) Let $w \in l^2(\mathbb{Z}_N)$. Prove that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if $|\hat{w}(n)| = 1$ for all $n \in \mathbb{Z}_N$.

   b) Suppose $M \in \mathbb{N}$, $N = 2M$ and $w \in l^2(\mathbb{Z}_N)$. Prove that $\{R_{2k} w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if $|\hat{w}(n)|^2 + |\hat{w}(n+M)|^2 = 2$, for $n = 0, 1, \ldots, M-1$.

P.T.O.

8. a) Give an example of a first stage Shannon wavelet basis with justification. Is your Shannon basis real valued ? Give reason.

   b) Suppose $M \in \mathbb{N}$, $N = 2M$ and $u \in l^2(\mathbb{Z}_N)$ such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(\mathbb{Z}_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first stage wavelet basis for $l^2(\mathbb{Z}_N)$.

9. Assume $N/2^p$ is an integer greater than 6, where p is some positive integer. Starting with the identity $\left(\cos^2\left(\frac{\pi n}{N}\right)+\sin^2\left(\frac{\pi n}{N}\right)\right)^5 = 1$, construct a first stage wavelet basis for $l^2(\mathbb{Z}_N)$.

Unit – II

10. a) Prove that the space $l^2(\mathbb{Z})$ is complete.

    b) Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta}\rangle = 0$ for all $n \in \mathbb{Z}$. Prove that $f(\theta) = 0$ a.e.

11. a) Define the Fourier transform on $l^2(\mathbb{Z})$ and the inverse Fourier transform on $L^2([-\pi, \pi))$. Prove that they are inverses of each other.

    b) Suppose $w \in l^1(\mathbb{Z})$ and $z \in l^2(\mathbb{Z})$. Prove that $(z * w)^\wedge(\theta) = \hat{z}(\theta)\,\hat{w}(\theta)$ a.e.

12. a) Suppose $u, v \in l^1(\mathbb{Z})$. Prove that $B = \{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a complete orthonormal set in $l^2(\mathbb{Z})$ if and only if the system matrix $A(\theta)$ is unitary for all $\theta \in [0, \pi)$.

    b) Prove that a bounded translation invariant linear transformation $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a convolution operator.

Unit – III

13. a) Define the convolution $f*g$ of $f, g : \mathbb{R} \to \mathbb{C}$. If $f, g \in L^2(\mathbb{R})$, prove that $f*g$ is bounded and $|f*g(x)| \le \|f\|\,\|g\|$ for all $x \in \mathbb{R}$.

    b) If $f, g \in L^1(\mathbb{R})$, prove that $f*g \in L^1(\mathbb{R})$ with $\|f*g\|_1 \le \|f\|_1\|g\|_1$.

    c) For $f, g \in L^2(\mathbb{R})$ and $x, y \in \mathbb{R}$, prove that $\langle R_x f, R_y g\rangle = \langle f, R_{y-x}g\rangle$.

14. a) Define a Lebesgue point of $f \in L^1(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, prove that almost every point of $\mathbb{R}$ is a Lebesgue point of f.

    b) Define $f \in L^1(\mathbb{R})$ by $f(x) = 1$ for $\frac{1}{n} < x < \frac{1}{n} + \frac{1}{2^n}$, $n = 1, 2, \ldots$ and $f(x) = 0$ for all other x. Prove that f is not continuous at $x = 0$ but 0 is a Lebesgue point of f.

15. a) Suppose $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Prove that $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi} = f(x)$ at every Lebesgue point x of f. Also deduce the uniqueness of the Fourier transform in $L^1(\mathbb{R})$.

    b) Suppose $f, g \in L^1(\mathbb{R})$ and $\hat{f}, \hat{g} \in L^1(\mathbb{R})$. Prove that $\langle \hat{f}, \hat{g}\rangle = 2\pi\langle f, g\rangle$. **(4×12=48)**
