K18P 0322

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree (Reg./Suppl./Imp.)
Examination, March 2018
(2014 Admission Onwards)
MATHEMATICS
MAT 4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks. **(4×3=12)**

1. Define level sets and graph of a function. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = -x_1^2 + x_2^2$. Sketch its level sets and graph of the function.

2. Find the two orientations on the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 4$.

3. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t)$ is orthogonal to $\dot{\alpha}(t) \; \forall \, t \in I$.

4. Find the Weingarten map of the cylinder $x_2^2 + x_3^2 = a^2$ in $\mathbb{R}^3$, $a \neq 0$.

5. Find the length of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$, where $\alpha(t) = (t^2, t^3)$, $I = [0, 2]$, $n = 1$.

6. Express a torus as a parametrized surface in $\mathbb{R}^4$.

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries 12 marks. (4×12=48)

UNIT – I

7. a) Establish the existence and uniqueness of the maximal integral curve of a smooth vector field $\mathbb{X}$ (on an openset $U \subseteq \mathbb{R}^{n+1}$) through P in U. 8

   b) Let $\mathbb{X}$ be a vector field $\mathbb{X}(P) = (P, X(P))$ where $X(x_1, x_2) = (-x_2, x_1)$. Find the integral curve of $\mathbb{X}$ through the point (1, 0). 4

8. a) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p equal to $[\nabla f(p)]^{\perp}$. 7

   b) Let $f : U \to \mathbb{R}$ be a smooth function. Let $\alpha : I \to U$ be an integral curve of ∇f then show that for each $t_0 \in I$, the function f is increasing faster along α at $\alpha(t_0)$ than along any other curve passing through $\alpha(t_0)$ with the same speed. 5

9. a) State and prove Lagrange multiplier theorem. 5

   b) Show that the extreme values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ where $a, b, c \in \mathbb{R}$ on the unit circle $x_1^2 + x_2^2 = 1$ are the eigen values of the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. 5

   c) Sketch the cylinder $f^{-1}(0)$ where $f(x_1, x_2, x_3) = x_1 - x_2^2$. 2

UNIT – II

10. a) Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Then prove that the Gauss map maps S onto the unit sphere S^n. 7

    b) Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $\mathbb{R}^3$. Show that α is a geodesic of S iff α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some $a, b, c, d \in \mathbb{R}$. 5

11. a) Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic in S with $\dot{\alpha} \neq 0$, then show that a vector field $\mathbb{X}$ tangent to S along α is parallel along α iff both $\|\mathbb{X}\|$ and the angle between $\mathbb{X}$ and $\dot{\alpha}$ are constant along α. 6

    b) Prove that the Weingarten map is self adjoint. 6

12. a) Prove that local parametrization of plane curves are unique upto reparametrization. 8

    b) Let $\alpha(t) = (x(t), y(t)), t \in I$ be a local parametrization of the oriented plane curve C. Show that $K \circ \alpha = (x'y'' - y'x'')/(x'^2 + y'^2)^{3/2}$. 4

UNIT – III

13. a) Let C be an oriented plane curve, then prove that there exists a global parametrization of C iff C is connected. 8

    b) Define an exact differential 1-form and show that the integral of an exact 1-form over a closed curve is always zero. 4

14. a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbb{V}$ be a unit vector in S_p, $p \in S$. Then prove that there exist an open set $V \subseteq \mathbb{R}^{n+1}$ containing P such that $S \cap N(\mathbb{V}) \cap V$ is a plane curve and the curvature at P of this curve is equal to the normal curvature. 7

    b) Let S be the hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ oriented by unit normal vector field $\mathbb{N}(P) = \left(P, \frac{-x_1}{\|P\|}, \frac{x_2}{\|P\|}, \frac{x_3}{\|P\|}\right)$ for all $(x_1, x_2, x_3) \in S$. Then find the normal curvature of S at $P = (0, 0, 1)$. 5

15. a) Let S be an n-surface is $\mathbb{R}^{n+1}$ and let $P \in S$ then prove that there exists an open set V about P is $\mathbb{R}^{n+1}$ and a parametrized n-surface $\phi : U \to \mathbb{R}^{n+1}$ such that ϕ is one to one map from U onto $V \cap S$. 6

    b) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $f : S \to \mathbb{R}^k$. Then prove that f is smooth iff $f \circ \phi : U \to \mathbb{R}^k$ is smooth for each local parametrization $\phi : U \to S$. 6

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