Reg. No. : ....

Name : ....

K17P 0405

Fourth Semester M.Sc. Degree (Reg./Supple./Imp.)
Examination, March 2017
(2014 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 60

***Instruction** : Answer **four** questions from Part **A**. **Each** question carries **3** marks. Answer **four** questions from Part **B** without omitting **any** Unit. **Each** question carries **12** marks.*

PART – A

1. Define eigen spectrum $\sigma_e(A)$ and approximate eigen spectrum $\sigma_a(A)$ of the operator A defined on the normed space X and prove that $\sigma_e(A) \subset \sigma_a(A)$.

2. Let X be a finite dimensional normed space. Prove that a sequence $x_n \to x$ in X weakly iff if and only it if converges to x in X.

3. Let X be a finite dimensional normed space. Prove that X is uniformly convex if and only if it is strictly convex.

4. Prove that CL(X, Y) the space of all compact linear maps from the normed space X into the normed space Y is subspace of BL(X, Y).

5. Let $H = e^2$ and $A(x(1), x(2)) = a\,x(1) + bx(2),\ c\,x(1) + dx(2)$ for $(x(1), x(2)) \in H$ and fixed a, b, c, d $\in \mathbb{C}$. Prove that A is normal iff $|b|^2 = |c|^2$ and $(a-b)\bar{c} = (\overline{a-d})b$.

6. Show by examples neither $\sigma(A)$ nor $\omega(A)$ is contained in the other in general, where A is a bounded linear operator on the Hilbert space H. **(4×3=12)**

P.T.O.

PART – B

Unit – I

7. a) Let $X = C([a, b])$ with the sup norm. For $x_0 \in X$ and for $x \in X$, let $Ax = x_0x$. Prove that A is bounded linear operator as X and find its spectrum.

b) Prove or disprove : Every bounded sequence in $\ell^{\prime}$ has a weak convergent subsequence.

8. a) State and prove spectral radius formula for an operator on a nonzero Banach space over $\mathbb{C}$.

b) Prove that every bounded sequence in X' has a weak * convergent subsequence provided X is a separable normed space.

9. a) Let X be a Banach space and $A \in BL(X)$. Let $k \in \mathbb{K}$ such that $|K|^p > \|A^p\|$ for some positive integer p. Then prove that $k \notin \sigma(A)$ and $(A - kI)^{-1} = -\sum_{n=0}^{\infty} \frac{A^n}{K^{n+1}}$

b) Let $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Show that the dual of $\mathbb{K}^n$ with $\|\cdot\|_p$ is linearly isometric to $\mathbb{K}^n$ with $\|\cdot\|_q$.

Unit – II

10. a) Show that every closed subspace of a reflexive normed space is reflexive.

b) Let X and Y be Banach spaces. Suppose $F : X \to Y$ is a compact linear map such that R(F) is closed in Y. Then prove that F is continuous and F has finite rank.

11. Let X and Y be normed spaces and $F \in BL(X, Y)$. If $F \in CL(X, Y)$ then prove that $F' \in CL(Y', X')$. Prove that the converse holds if Y is a Banach space.

12. a) Let X be an infinite dimensional normed space and $A \in CL(X)$. Prove that $\sigma_a(A)$ non empty.

b) Prove that the eigen spectrum of a compact linear operator on a normed space X is countable.

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Prove that A is onto if and only if A^* is bounded below.

b) Let E be a measurable subset of $\mathbb{R}$ and $H = L^2(E)$. Fix $Z \in L^\infty(E)$ and define $Ax = 2x$, $x \in H$. Find the adjoint A^* of A.

14. a) Let H be separable Hilbert space and $u_1, u_2, \ldots$ constitute an orthonormal basis for H. Let (k_n) be a bounded sequence of scalars and

$$A_x = \sum_n k_n \langle x, u_n \rangle u_n, \; x \in H.$$

Then prove that $A \in BL(H)$ and A is normal. Also find conditions on (k_n) such that A is unitary and self adjoint.

b) Let A be a self adjoint operator on a Hilbert space H. Prove that A = 0 if $\langle A_x, x \rangle = 0$ for all $x \in H$.

c) Define a compact operator on a Hilbert space H.

15. a) Let H be a separable Hilbert space. Let $A \in BL(H)$ be a Hilbert-Schmidt operator. Prove that A is compact.

b) Let A be a compact operator on a non zero Hilbert space H. Prove that every non zero approximate eigen value of A is an eigen value of A. Also prove that the eigen space corresponding to a non zero eigen value, is finite dimensional.

(4×12=48)