K16P 0197

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree (Regular/Supplementary/Improvement) Examination, March 2016
MATHEMATICS (2014 Admission)
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 60

Instructions : *1) Answer **four** questions from Part – **A**. **Each** question carries **3** marks.*

*2) Answer **four** questions from Part – **B** without omitting **any** Unit. **Each** question carries **12** marks.*

PART – A

1. Define the transpose F′ of a bounded linear map F from a normed space X into a normed space Y and prove that $F''J_X = J_Y F$, where $J_X : X \to X''$ and $J_Y : Y \to Y''$ are the canonical embeddings.

2. Define weak convergence of a sequence (x_n) in a normed space X. Prove that every convergent sequence in any normed space converges weakly. What about the converse ? Justify your claim.

3. Prove that every compact linear map from a normed space X into a normed Y is continuous. Show by an example the converse is in general is not true.

4. Prove that every non zero spectral value of a compact linear operator in a normed space is its eigen value.

5. Let $A \in BL(H)$ and $K_l \neq K_L \in IK$. If $Ax_1 = K_1 x_1$ and $A^+x_2 = \overline{K}_2 x_2$ for x_1 x_2 in the Hilbert space H, then prove that $x_1 \perp x_2$.

6. Prove that the adjoint of any compact operator defined on any Hilbert space H is compact. (4×3=12)

P.T.O.

PART – B

Unit – I

7. a) Let X be a Banach space, $A \in BL(X)$ and $\| A^p \| < 1$ for some positive integer p. Prove that $(I - A)$ is invertible.

   b) Let X be a normed space such that its dual X′ is separable. Then prove that X is separable.

8. a) Prove that the set of all invertible operators is open in BL(X), where X is a normed space.

   b) Let (x_n^1) be a sequence in a Banach space X. Prove that (x_n^1) is bounded and $(x_n^1(x))$ is a Cauchy sequence in $\mathbb{K}$ for each x in a subset of X whose span is dense in X if and only if (x_n^1) is weak* convergent in X′.

9. a) Let $X = l^p$ with $\| - \|_p$, $1 \leq p \leq \infty$. For $x = (x(1), x(2) \ldots) \in X$, let $Ax = \left(x(1), \frac{x(2)}{2}, \frac{x(3)}{3} \ldots\ldots \right)$. Prove that $A \in BL(X)$ and $\| A \|_p = 1$. Also find the spectrum of A.

   b) Let X be a separable normed space. Prove that every bounded sequence in X′ has a weak* convergent subsequence.

Unit – II

10. a) Let X be a non zero reflexive space. If f is a continuous linear functional on X prove that there is some $x_1 \in X$ such that $\| x_1 \| = 1$ and $| f(x_1) | = \| f \|$.

    b) Let X be a normed space, $A : X \to X$ be linear and $Ax_n = K_n x_n$ for some $0 \neq x_n \in X$ and $K_n \in \mathbb{K}$, $n = 1, 2 \ldots$ If $K_n \neq K_m$ whenever $n \neq m$, then prove that

       i) $\{x_1, x_2 \ldots\}$ is a linearly independent subset of X.

       ii) $K_n \to 0$ provided A is compact and $\{x_1, x_2 \ldots\}$ is infinite.

11. Show that if X is a Banach space then CL(X) is a closed ideal of BL(X).

12. a) Let X be a reflexive space. Prove that every bounded sequence in X has a weak convergent subsequence.

    b) Let X and Y be normed spaces. If $F \in CL(X, Y)$ then prove that $F' \in CL(Y', X')$.

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle Ax, y \rangle = \langle x, By \rangle$.

    b) Let H be a separable Hilbert space and $u_1, u_2 \ldots\ldots$ constitute an orthonormal basis for H. If $A \in BL(H)$ is defined by the matrix $M = (K_{iy})$, with respect to the orthonormal basis $\{u_1\ u_2 \ldots\}$ for H then prove that the matrix $(\overline{K_{ji}})$ defines the operation A* and deduce that the adjoint of the right shift operator on l^2 is the left shift operator on l^2.

14. a) Let H be a Hilbert space and $A \in BL(H)$. If A is self adjoint then prove that $\| A \| = \text{Sup}\{| \langle A x, x \rangle | : x \in H, \| x \| \leq 1\}$

    b) Let H be a Hilbert space and $\mathbb{K} = \mathbb{C}$. Prove that there are unique self adjoint operators B and C on H such that $A = B + iC$.

    c) Let A be a normal operator on the Hilbert space H. Prove that if $K \in \sigma_e(A)$ then $\overline{K} \in \sigma_e(A^*)$ and an eigen vector of A corresponding to K is also an eigen vector of A* corresponding to $\overline{K}$. Show by an example this result may not hold for an arbitrary operator on H.

15. For a bounded linear operator on a Hilbert space H define W(A), the numerical range of A. Show that $\sigma(A)$ contained in the closure of W(A). and show by examples neither $\sigma(A)$ nor W(A) is contained in the other. **(4x12=48)**