K16P 0198

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree (Regular/Supplementary/Improvement) Examination, March 2016

(2014 Admission)

MATHEMATICS

MAT 4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks :

1. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F\ \mathbb{R}^{n+1} \to \mathbb{R}$.

2. Let $f : U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f. Show that $\frac{d}{dt}(f \circ \alpha)(t) = \|\nabla f(\alpha(t))\|^2$ for all t in I.

3. Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, t)$.

4. Compute $\nabla_{\mathbf{v}} X$ where $\mathbf{v} \in \mathbb{R}^2_p$, $p \in \mathbb{R}^2$ and X are given by $X(x_1, x_2) = (x_1, x_2, x_1 x_2, x_2^2)$, $\mathbf{v} = (1, 0, 0, 1)$.

5. Find the length of the parametrized curve $\alpha : [-1, 1] \to \mathbb{R}^3$ defined by $\alpha(t) = (\cos 3t, \sin 3t, 4t)$.

6. Obtain the torus as a parametrized 2-surface in $\mathbb{R}^4$. **(4×3=12)**

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **12** marks :

UNIT – I

7. a) Let $\mathbb{X}$ be a vector field on $\mathbb{R}^2$: $\mathbb{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (-x_2, x_1)$. Find the integral curve of $\mathbb{X}$ through $p = (a, b)$.

   b) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is $[\nabla f(p)]^{\perp}$.

   c) Sketch the level set $f^{-1}(0)$ and typical values $\nabla f(p)$ of the vector field ∇f for $p \in f^{-1}(0)$, where $f(x_1, x_2) = x_1^2 + x_2^2 - 1$.

8. a) Define an n-surface in $\mathbb{R}^{n+1}$. Give an example with justification.

   b) State and prove Lagrange multiplier theorem.

   c) Express the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$ as level set of two different functions.

9. a) Establish the existence and uniqueness of integral curves of a smooth tangent vector field $\mathbb{X}$ on an n-surface S in $\mathbb{R}^{n+1}$ through p in S.

   b) Prove that every connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

   c) Find the two orientations of the unit 1-sphere $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^2$.

UNIT – II

10. a) Let S be a compact connected oriented n surface in $\mathbb{R}^{n+1}$ with $S = f^{-1}(c)$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function such that $\nabla f(p) \neq 0$ for all $p \in S$. Prove that the Gauss map $N : S \to S^n$ is onto.

    b) Show that the spherical image of an n-surface with orientation $\mathbb{N}$ is the reflection through the origin of the spherical image of the same n-surface with orientation $\mathbb{N}$.

11. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$ and let $v \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field $\mathbb{V}$ tangent to S along α which is parallel and has $\mathbb{V}(t_0) = v$.

    b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve, let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields tangent to S along α. Show that i) $(\mathbb{X} + \mathbb{Y})' = \mathbb{X}' + \mathbb{Y}'$ and $(f\mathbb{X})' = f'\mathbb{X} + f\mathbb{X}'$, for all smooth functions f along α.

12. a) Prove that the Weingarten map is self adjoint.

    b) Prove that local parametrization of a plane curve is unique upto reparametrization.

    c) Find the curvature of the curve $(x_1 - a)^2 + (x_2 - b)^2 = r^2$, $r > 0$ oriented by the inward normal vector field.

UNIT – III

13. a) Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrization of C. Prove that β is either one to one or periodic.

    b) Define a differential 1-form. When is said to be exact ? Give an example (with justification) of a 1-form on $\mathbb{R}^2 - \{0\}$ which is not exact.

14. a) For each compact oriented n-surface S in $\mathbb{R}^{n+1}$, prove that there exists a point p such that the second fundamental form at p is definite.

    b) Let S be the hyperboloid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$, where a, b and c all $\neq 0$. Find the Gaussian curvature $K : S \to \mathbb{R}$.

15. a) Define differential of a smooth map. Show that it does not depend on the choice of the parametrized curve α.

    b) Let Φ be the parametrized torus in $\mathbb{R}^3$ $\Phi(\theta, \phi) = ((a + b\cos\phi)$, $\cos\theta, (a + b\cos\phi)\sin\theta, \sin\phi)$. Find the Gaussian curvature of Φ.

    c) State (no proof) inverse function theorem for n – surfaces. **(4×12=48)**

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