14. a) If $P(x) > 0$, $a \leq x \leq b$ and if $[a, b]$ contains no points conjugate to a, then prove that the quadratic functional $\int_a^b (Ph'^2 + Qh^2)dx$ is positive definite for all $h(x)$ with $h(a) = h(b) = 0$.

b) If the extremal $y = y(x)$ corresponds to a minimum of the functional $\int_a^b F(x, y, y')dx$ and if $F_{y'y'} > 0$ along this extremal, then prove that (a, b) contains no points conjugate to a.

15. a) For $y = (y_1, \ldots, y_n)$, $y' = (y_1', \ldots, y_n')$, express the second variation of $J[y] = \int_a^b F(x, y, y')dx$ in the form $\int_a^b [(Ph', h') + (Qh, h)]dx$ for suitable matrices P and Q.

b) If P is a positive definite symmetric matrix and if $[a, b]$ contains no points conjugate to a then show that the quadratic functional $\int_a^b [(Ph', h') + (Qh, h)]dx$ is positive definite for all $h(x)$ such that $h(a) = 0 = h(b)$. **(4×12=48)**

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree (Regular/Supplementary/Improvement) Examination, March 2016

MATHEMATICS

(2014 Admission)

MAT4E03 : Calculus of Variation

Time : 3 Hours Max. Marks : 60

PART – A

Answer **four** questions from this Part. **Each** question carries **3** marks.

1. Find the shortest curve joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

2. Find the extremals of the functional $\int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2)dx$.

3. Find the transversality condition for the functional

$$J[y] = \int_{x_0}^{x_1} f(x, y)\sqrt{1 + y'^2}\, e^{\tan^{-1} y'}\, dx.$$

Interpret the conditions geometrically.

4. Show that the functional $J[y] = \int_{x_0}^{x_1} y'^2\, dx$ is invariant under the transformation $x^* = x + \varepsilon$, $y^* = y$, where ε is an arbitrary constant.

5. Calculate the second variation of the functional $e^{J[y]}$, where $J[y]$ is a twice differentiable functional.

6. State a set of sufficient conditions for a functional $J[y] = \int_a^b F(x, y, y')dx$, $y(a) = A$, $y(b) = B$ to have a weak minimum for the curve $y = y(x)$. **(4×3=12)**

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries 12 marks.

Unit – I

7. a) Define differential of a functional and prove that the differential of a differentiable functional is unique.

   b) Prove that a necessary condition for the differentiable functional J[y] to have an extremum for $y = \hat{y}$ is that its $\delta J[b] = 0$ for $y = \hat{y}$ and all admissible h.

8. a) Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the x-axis.

   b) Establish the invariance of Euler's equation under a transformation of the form x = x (u, v) y = y (u, v) for which $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0$.

   c) Use the transformation $x = r\cos\phi$, $y = r\sin\phi$ to find the extremals of $J[r] = \int_{\phi_0}^{\phi_1} \sqrt{r^2 + r'^2}\, d\phi$, where $r = r(\phi)$.

9. a) Given the functional $J[y] = \int_a^b F(x, y, y')\,dx$, let the admissible curves satisfy the conditions y (a) = A, y (b) = B, $K[y] = \int_a^b G(x, y, y')\,dx = l$, where K [y] is another functional, and let J[y] have an extremum for y = y (x). If y = y (x) is not an extremal if K [y], prove that there exists a constant λ such that y = y (x) is the extremal of the functional $\int_a^b (F + \lambda G)\,dx$.

   b) Among all curves of length *l* in the upper half plane passing through the points (– a, 0) and (a, 0), find the one which together with the interval [– a, a] encloses the largest area.

Unit – II

10. a) Derive the basic formula for the general variation of the functional.

$$J[y_1, y_2, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\,dx.$$

   b) Find the curve for which the functional $J[y] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx$, y (0) = 0 can have extrema if the point (x_1, y_1) can vary along the circle $(x - 9)^2 + y^2 = 9$.

11. a) Prove that a necessary and sufficient condition for $\phi = \phi(y_1, \ldots, y_n, p_1, \ldots, p_n)$ to be a first integral of the system of Euler equations is that the Poisson bracket $[\phi, H]$ vanish identically.

   b) Write and solve the Hamilton-Jacobi equation corresponding to the functional $J[y] = \int_{x_0}^{x_1} \sqrt{x^2 + y^2}\,\sqrt{1+y'^2}\,dx$.

12. a) State and prove the principle of least action.

   b) Deduce the laws of conservation of energy and momentum.

Unit – III

13. a) Prove that a necessary condition for the functional J[y] to have a minimum for $y = \hat{y}$ is that $\delta^2 J[y] \geq 0$ for $y = \hat{y}$ and all admissible h.

   b) Show that the second variation for the functional $J[y] = \int_a^b F(x, y, y')\,dx$ defined for curves y = y(x) with fixed end points y (a) = A, y (b) = B can be expressed as $\delta^2 J[h] = \int_a^b (Ph'^2 + Qh^2)\,dx$.