9. a) Prove that a necessary condition for $J[y] = \int_a^b F(x, y, y')\,dx$, $y(a) = A$, $y(b) = B$ to have a minimum for the curve $y = y(x)$ is that $F_{y'y'} \ge 0$ at every point of the curve.

b) If the extremal $y = y(x)$ corresponds to a minimum of the functional $\int_a^b F(x, y, y')\,dx = 0$ and if $F_{y'y'} > 0$ along this extremal then prove that (a, b) contains no points conjugate to a .

10. a) If P is a positive definite symmetric matrix and if [a, b] contains no points conjugate to a, then prove that the quadratic functional $\int_a^b [(P\,h', h') + (Q\,h, h)]\,dx$ is positive definite for all h(x) such that h(a) = h(b) = 0.

b) Show that the extremals of the functional $\int_a^b F(x, y, y')\,dx$ have no conjugate points. **(4×12=48)**

Reg. No. : ....

Name : ....

IV Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, March 2015

MATHEMATICS

Elective : Calculus of Variations

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. a) Find the extremals of the functional $\int_a^b \left(y^2 + y'^2 - 2y \sin x\right) dx$.

b) Prove by direct calculation that the isosceles triangle has the greatest area among all triangles with a given base and given perimeter.

c) Prove that one and only one extremal of the functional $\int e^{-2y^2}\left(y'^2 - 1\right) dx$ passes through any two points of the plane with different abscissas.

d) Find the transversality conditions for the functional $J[y] = \int_{x_0}^{x_1} f(x, y)\sqrt{1 + y'^2}\,dx$.

e) Find the canonical equations of the functional $\int_a^b \left(P\,y'^2 + Qy^2\right) dx$, where P and Q are functions of x.

f) Find the second variation of the functional $e^{J[y]}$, where J[y] is a twice differentiable function. **(4×3=12)**

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **12** marks.

UNIT – I

2. a) Explain the concept of variation of a functional J[y]. Prove that differential of a differentiable functional is unique.

   b) Prove that a necessary condition for the differentiable functional J[y] to have an extremum for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$.

   c) Let $J[y] = \int_a^b F(x, y, y')\,dx$ be defined on the set of function y(x) with continuous first derivative in [a, b] and y(a) = A, y(b) = B. Find a necessary condition for J[y] to have an extremum.

3. a) Among all curves whose end points lie on two given vertical lines x = a and x = b, find the curve for which $J[y] = \int_a^b F(x, y, y')\,dx$ has an extremum.

   b) Starting from the point (a, A), a heavy particle slides down a curve in the vertical plane. Find the curve such that the particle reaches the vertical line x = b (b ≠ a) in the shortest time.

4. a) Let $J[y] = \int_a^b F(x, y, y')\,dx$ where y = y(x) satisfy y(a) = A, y(b) = B, $K[y] = \int_a^b G(x, y, y')\,dx = l$. Let J[y] have an extremum for y = y(x). Prove that if y = y(x) is not an extremal of K[y], then there is a constant λ such that y is an extremal of $\int_a^b (F + \lambda G)\,dx$.

   b) Among all curves lying on the sphere $x^2 + y^2 + z^2 = a^2$ and passing through two given points (x_0, y_0, z_0) and (x_1, y_1, z_1) find the one which has least length.

UNIT – II

5. a) Derive Weierstrass-Erdmann corner conditions in the context of a weak extremum of $\int_a^b F(x, y, y')\,dx$ where the admissible functions are continuous for a ≤ x ≤ b except possibly at some point C (a < c < b) with y(a) = A, y(b) = B.

   b) Find the extremals of the functional $J[y] = \int_0^4 (y'-1)^2 (y'+1)^2\,dx$, y(0) = 0, y(4) = 2 which have just one corner.

6. a) Derive the canonical system of Euler equations for

   $$J[y_1, y_2, \ldots, y_n] = \int_a^b F(x, y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n')\,dx.$$

   b) Prove that a necessary and sufficient condition for a function $\Phi = \Phi(y_1, \ldots, y_n, p_1, \ldots, p_n)$ to be a first integral of the system of Euler equations in that the Poisson bracket [Φ, H] vanish identically.

7. a) State and prove Noether's theorem on the invariance of the functional $\int_{x_0}^{x_1} F(x, y, y')\,dx$ under the family of transformations. $x^* = \Phi(x, y, y'; \varepsilon)$, $y_i^* = \psi_i(x, y, y'; \varepsilon)$, i = 1, 2, ..., n.

   b) State and prove the principle of least action.

UNIT – III

8. a) Prove that a necessary condition for a functional J[y] to have a minimum for $y = \hat{y}$ is that $\delta^2 J[y] \geq 0$ for $y = \hat{y}$ and all admissible h.

   b) For $J[y] = \int_a^b F(x, y, y')\,dx$ defined for curves y = y(x) with y(a) = A, y(b) = B derive an expression for $\delta^2 J[h]$ and show that it can be written as $\int_a^b (P h'^2 + Q h^2)\,dx$.