14. a) If $P(x) > 0$ $(a \le x \le b)$, and if the interval [a, b] contains no points conjugate to a then prove that the quadratic functional $\int_a^b \left(P h'^2 + Q h^2\right) dx$ is positive definite for all h(x) such that h(a) = h(b) = 0.

b) Establish Jacobi's necessary condition for the extremal y = y(x) corresponds to a minimum of the functional $\int_a^b F(x, y, y')\, dx$, $y(a) = A$, $y(b) = B$.

15. a) If P is a positive definite symmetric matrix and if the interval [a, b] contains no points conjugate to a then prove that the quadratic functional

$$\int_a^b [(P h', h') + (Q h, h)]\, dx$$ is positive definite for all h(x) such that

h(a) = h(b) = 0.

b) Obtain the second variation of the functional $e^{J[y]}$, where J[y] is a twice differentiable functional. **(4×12=48)**

Reg. No. : ....

Name : ....

IV Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, March 2014

MATHEMATICS

Paper – XVII : Elective : Calculus of Variations

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. Find the extremals of the functional

$$J[r] = \int_{\varphi_0}^{\varphi_1} \sqrt{r^2 + r'^2}\, d\varphi, \text{ where } r = r(\varphi).$$

2. Find the extremals of a functional of the form $\int_{x_0}^{x_1} F(y', z')\, dx$, given that

$$F_{y'y'}F_{z'z'} - (F_{y'z'})^2 \ne 0 \text{ for } x_0 \le x \le x_1.$$

3. Find the curves for which the functional

$$J[y] = \int_0^{\pi/4} \left(y^2 - y'^2\right) dx$$ can have extrema, given that y(0) = 0, while the right hand end point can vary along the line $x = \pi/4$.

4. Prove that the functional $J[y] = \int_{x_0}^{x_1} y'^2\, dx$ is invariant under the transformation $x^* = x + \varepsilon$, $y^* = y$, where ε is an arbitrary constant.

5. Find the Hamilton-Jacobi equation corresponding to the functional

$$J[y] = \int_{x_0}^{x_1} \sqrt{\varphi_1(x) + \varphi_2(y)}\, \sqrt{1 + y'^2}\, dx.$$

6. State three necessary conditions for the functional $\int_a^b F(x, y, y')\, dx$, $y(a) = A$, $y(b) = B$ to have a weak extremum for the curve $y = y(x)$. **(4x3=12)**

PART – B

Answer **any four** questions without omitting any Unit. **Each** question carries **12** marks.

UNIT – I

7. a) Define the variation $\delta J[h]$ of a functional $J[y]$ and prove that a necessary condition for the differentiable functional $J[y]$ to have an extremum for $y = \hat{y}$ is that $\delta J[h] = 0$ for $y = \hat{y}$ and all admissible h.

b) Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the x-axis.

8. a) Discuss the method of solving the variable end point problem : Among all curves whose end points lie on two given vertical lines x = a and x = b, find the curve for which the functional $J[y] = \int_a^b F(x, y, y')\, dx$ has an extremum.

b) Find the curve joining two fixed points A and B traversed by a particle sliding under gravity from A to B in least time.

9. a) Derive a necessary condition for an extremum of a functional

$$J[y_1, ..., y_n] = \int_a^b F\left(x, y_1, ..., y_n, y_1', ..., y_n'\right) dx$$ which depends on n continuously differentiable functions $y_1(x), ..., y_n(x)$ satisfying the boundary conditions $y_i(a) = A_i$, $y_i(b) = B_i$ (i = 1, 2, ..., n).

b) Obtain the differential equations for the curves along which the light propagates in an inhomogeneous medium.

UNIT – II

10. a) Determine the transversality conditions for the problem : Among all smooth curves whose end points P_0 and P_1 lie on two given curves $y = \varphi(x)$ and $y = \psi(x)$, find the curve for which $J[y] = \int_{x_0}^{x_1} F(x, y, y')\, dx$ has an extremum. Also derive the transversality conditions for functionals of the forms $\int_{x_0}^{x_1} f(x, y)\sqrt{1 + y'^2}\, dx$.

b) Obtain the Weierstrass-Erdmann corner conditions for an extremal with corners of the functional $J[y] = \int_a^b F(x, y, y')\, dx$.

11. a) Obtain the canonical systems of Euler equations for the functional

$$J[y_1, ..., y_n] = \int_a^b F\left(x, y_1, ..., y_n, y_1', ..., y_n'\right) dx.$$

b) Prove that a necessary and sufficient condition for a function $\Phi = \Phi(y_1, ..., y_n, p_1, ..., p_n)$ to be the first integral of the systems of Euler equations is that the Poisson bracket $[\Phi, H]$ vanish identically.

12. a) State and prove the principle of least action.

b) Deduce the laws of conservation of energy and momentum.

UNIT – III

13. a) Prove that a necessary condition for the functional J[y] to have a minimum for $y = \hat{y}$ is that $\delta^2 J[y] \geq 0$ for $y = \hat{y}$ and all admissible h.

b) For the functional $J[y] = \int_a^b F(x, y, y')\, dx$, $y(a) = A$, $y(b) = B$, prove that the second variation

$$\delta^2 J[h] = \int_a^b \left(P h'^2 + Q h^2\right) dx.$$