0150708 **K19P 1187**

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS-Reg./Suppl./Imp.) Examination, October - 2019

(2017 Admission Onwards)

MATHEMATICS

MAT 3C 12 : FUNCTIONAL ANALYSIS

Time : 3 Hours Max. Marks : 80

PART - A

Answer **Four** questions from this part. Each question carries **4** marks. **(4×4=16)**

1. Show that if $x_n \to x$ in l^2 then $x_n \to x$ in l^∞.
2. Give an example of an element in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$ and prove your claim.
3. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $K^n, n=1,2,...$ are equivalent.
4. Show that c_0 is a Banach space.
5. Show that the inverse of a bijective continuous map may not be continuous.
6. Among l^p spaces, $1 \le p \le \infty$, select the Hilbert spaces and prove your claim.

PART - B

Answer **4** questions from this part without omitting any unit. Each question carries **16** marks. **(4×16=64)**

UNIT - I

7. a) Let $\|\cdot\|_j$ be a norm on a linear space $X_j, j=1,2,...,m$. Fix p such that $1 \le p \le \infty$. Fix $x=(x(1),...,x(m))$ in the product space

   $X = X_1 \times ... \times X_m$, let $\|x\|_p = (\|x(1)\|_1^p + ... + \|x(m)\|_m^p)^{\frac{1}{p}}$,

   If $1 \le p < \infty$ and $\|x\|_\infty = \max\{\|x(1)\|_1, ..., \|x(m)\|_m\}$ Then show that $\|\cdot\|_p$ is a norm on X. Also show that a sequence (x_n) converges to x in X if and only if $(x_n(j))$ converges to x(j) in X_j for every j=1,..., m.

P.T.O.

b) Let X be a normed space. Then show that the following are equivalent.

i) Every closed and bounded subset of X is compact.

ii) The subset $\{x \in X : \|x\| \leq 1\}$ of X is compact.

iii) X is finite dimensional.

8. a) Let X and Y be normed space and $F : X \to Y$ be a linear map. Then show that the following conditions are equivalent.

i) F is bounded on $U(0,r)$ for some $r > 0$.

ii) F is continuous at 0.

iii) F is continuous on X.

iv) F is uniformly continuous on X.

v) $\|F(x)\| \leq \alpha\|x\|$ for all $x \in X$ and some $\alpha > 0$.

vi) The zero space Z(F) of F is closed in X and the linear map $\tilde{F} : X/Z(F) \to Y$ defined by $\tilde{F}(x + Z(F)) = F(x), x \in X$ is continuous.

b) Define bounded linear map and operator norm.

9. a) State and prove Taylor-Foguel Theorem.

b) Show that a Banach space cannot have a denumerable basis.

UNIT - II

10. a) State and prove Uniform boundedness principle.

b) Let X be a normed linear space and (x_n) be a sequence in X. Prove or disprove: (x_n) converges in X if and only if $f(x_n)$ converges in K for every $f \in X'$.

11. a) Prove of disprove : The inverse of a bijective continuous map is continuous.

b) Let X be a linear space over K. Consider subsets U and V of X, and $k \in K$ such that $U \subset V + kU$ Then show that every $x \in U$, there is a sequence $\upsilon_n)$ in V such that $x - (u_1 + ku_2 ... + k^{n-1}u_n) \in k^n U, n = 1, 2...$

c) Define projection operator and give an example.

12. a) State and prove open mapping theorem.

b) Show that the closed graph theorem may not hold if the range of the linear map is not a Banach space.

UNIT - III

13. a) State and prove Bessel's inequality.

b) Let X be an inner product space, $\{u_1, u_2 ...\}$ be a countable orthonormal set in X and $k_1, k_2, ...$ belong to K. if X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then prove that $\sum_n k_n u_n$ converges in X.

14. a) State and prove Riesz representation Theorem.

b) What do you mean by weak convergence?

15. a) Let H be a Hilbert space. For $y \in H$, define $j_y : H' \to K$ by $j_y(f) = f(y), f \in H'$. Then prove that j_y is a continuous linear functional on H' and the map J from H to H'' defined by $J(y) = j_y, y \in H$, is a surjective linear isometry.

b) If the underlying space is a Hilbert space then show that Hahn-Banach extension is unique.