15. a) Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E and $f'(x) = 0$ for all $x \in E$, then prove that f is constant.

b) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then prove that $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

0151498 **K19P 1189**

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS-Reg./Suppl./Imp.)
Examination, October - 2019
(2017 Admn. Onwards)
MATHEMATICS
MAT3C14 : ADVANCED REAL ANALYSIS

Time : 3 Hours Max. Marks : 80

PART - A

Answer **Four** questions from this part. Each question carries 4 marks. **(4×4=16)**

1. If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions and converge uniformly on a set E, prove that $\{f_n g_n\}$ converges uniformly on E.

2. Consider $f(x)=\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. Is f continuous wherever the series converges?

3. Show that e^x defined on $\mathbb{R}^1$ satisfy the relation $e^{x+y} = e^x e^y$.

4. Show that the functional equation $\Gamma(x+1) = x\Gamma(x)$ holds if $0 < x < \infty$.

5. Prove that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

6. If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then prove that $\|BA\| \le \|B\| \|A\|$.

PART - B

Answer **Four** questions from this part without omitting any unit. Each question carries **16** marks. **(4×16=64)**

UNIT - I

7. a) Suppose $\lim_{n\to\infty} f_n(x) = f(x) (x \in E)$ and put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$.

Show that $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

b) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n$ $(n = 1,2,...)$. Then prove that $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

8. a) If X is a metric space, $C(X)$ denote the set of all complex valued, continuous, bounded functions with domain X. Show that $C(X)$ with supremum norm is a metric space.

b) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

9. a) If K is a compact metric space, if $f_n \in C(K)$ For $n = 1,2,..$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K then prove that

i) $\{f_n\}$ is uniformly bounded on K.

ii) $\{f_n\}$ contains a uniformly convergent subsequence.

b) Define equicontinuity and give an example.

UNIT - II

10. a) Suppose the series $\sum_{n=0}^{\infty} C_n x^n$ converges for $|x|<R$ and define $f(x) = \sum_{n=0}^{\infty} C_n x^n \left(|x| < R\right)$.

Then prove that $\sum_{n=0}^{\infty} C_n x^n$ converges unifromly on $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon > 0$ is chosen. Also shows that the funciton f is continuous and differentiable in $(-R,R)$, and $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} ((|x| < R)$.

b) Given a double sequence $\{a_{ij}\}$, i=1,2,... j=1,2,... suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i \, (i = 1,2,...)$ and $\sum b_i$ converges. Then show that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

11. a) Suppose $a_0, ..., a_n$ are complex numbers $n \geq 1, a_n \neq 0, P(z) = \sum_{k=0}^{n} a_k z^k$. Then prove that P(z)=0 for some complex number z.

b) If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that $|f(x+t) - f(x)| \leq M|t|$ for all $t \in (-\delta, \delta)$, then prove that $\lim_{N\to\infty} S_N(f;x) = f(x)$.

12. a) If f is continuous (with period 2π) and if $\varepsilon > 0$, then prove there is a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all real x.

b) If f is a positive function on $(0,\infty)$ such that

i) $f(x+1) = x f(x)$.

ii) $f(1) = 1$

iii) log f is convex

Then prove that $f(x) = \Gamma(x)$.

UNIT - III

13. a) Suppose X is a vector space, and dim X=n. Show that

i) a set E of n vectors in X spans X if and only if E is independent

ii) X has a basis, and every basis consist of n vectors.

iii) If $1 \leq r \leq n$ and $\{y_1, y_2, ..., y_r\}$ is an independent set in X, then show that X has a basis containing $\{y_1, y_2, ..., y_r\}$.

b) Define linear transformation and give an example.

14. a) Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. If $A \in \Omega, B \in L(\mathbb{R}^n)$, and $\|B - A\| . \|A^{-1}\| < 1$, then prove that $B \in \Omega$.

b) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is dirrerentiable at $f(x_0)$. Then prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0) f'(x_0))$.