Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, November 2017
(2014 Admn. Onwards)
MATHEMATICS
MAT3C14 : Advanced Real Analysis

Time : 3 Hours | Max. Marks : 60

Instructions : *1) Answer **any four** questions from Part **A**. **Each** question carries **3** marks.*
*2) Answer **any four** questions from Part **B** without omitting **any** Unit. **Each** question carries **12** marks.*

PART – A

1. Prove that every uniformly convergent sequence of bounded functions from the closed interval [a, b] of $\mathbb{R}$ into $\mathbb{R}$ is uniformly bounded.
2. Show by an example that a convergent series of continuous functions may have a discontinuous sum.
3. Define gamma function and prove that $\Gamma(n+1) = n!$ for n = 1, 2, 3....
4. Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x E^n$. If E has a limit point in S then prove that $a_n = b_n$ for n = 0, 1, 2,....
5. Prove or disprove 'If both the partial derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ exist at a point of $\mathbb{R}^2$ then f is continuous at that point'.
6. Define the derivative of a map $f : E \to \mathbb{R}^m$, where E is an open subset of $\mathbb{R}^n$, at a point $x \in E$ and prove that every linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

PART – B

Unit – I

7. a) If $\{f_n\}$ is a sequence continuous complex functions on the subset E. A metric space X and if $f_n \to f$ uniformly on E then prove that f is continuous.
   b) Prove that the space $\mathscr{C}(x)$ of all complex valued, continuous bounded functions on the metric space x with the metric induced by the supemum norm on $\mathscr{C}(x)$ is a complete metric space.

P.T.O.

8. a) Let $f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n}$, for m = 1, 2,.... and for $a \le x \le b$ $(a, b \in \mathbb{R})$. Prove that $\{f_m(x)\}$ converges to a function f on [a, b] which is everywhere discontinuous on [a, b] and which is not Riemann integrable.

b) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

9. If k is compact, if $f_n \in \mathscr{C}(k)$ for n = 1, 2, ... and if $\{f_n\}$ is pointwise bounded and equicontinuous on k then prove that

a) $\{f_n\}$ is uniformly bounded on k

b) $\{f_n\}$ contains a uniformly convergent subsequence.

Unit – II

10. a) Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$, the series converges in $|x| < R$, where $R > 0$.

If $-R < a < R$ then prove that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ $(|x-a| < R - |a|)$.

b) Prove that the complex field is algebraically closed.

11. a) Let $\{\phi_n\}$ be orthonormal on [a, b]. Let $s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$ be the n^{th} partial sum of the Fourier series of f and suppose $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$.

Then prove that $\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx$.

b) If $x > 0$, $y > 0$, prove that the beta function B (x, y) satisfies the identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

12. State and prove Parseval's theorem for Riemann integrable 2π periodic functions.

Unit – III

13. a) Prove that the set of all linear operators on $\mathbb{R}^n$ forms an open subset of $L(\mathbb{R}^n)$.

b) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing f(E) into $\mathbb{R}^k$ and g is differentiable at $f(x_0)$. Prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable and x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.

14. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and f is differentiable at a point $x \in E$, then prove that the partial derivatives $(D_j f_i)(x)$ exist for $i \le j \le n$ and $i \le j \le m$ and $f'(x) e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i$, $i \le j \le n$.

where $\{e_1, e_2, ... e_n\}$ and $\{u_1, u_2 .... u_n\}$ are standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

b) Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E and there is a constant M such that $\|f'(x)\| \le M$ for every $x \in E$. Prove that $|f(b) - f(a)| \le M|b - a|$, for all $a, b \in E$.

15. State and prove the inverse function theorem.
