Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg./Supple./Imp.) Examination, November 2017
(2014 Admn. Onwards)
MATHEMATICS
MAT 3C12 : Functional Analysis

Time : 3 Hours Max. Marks: 60

Instructions : 1) *Answer **any four** questions from Part **A**. **Each** question carries **3** marks.*

2) *Answer **any four** questions from Part **B** without omitting **any** Unit. **Each** question carries **12** marks.*

PART – A

1. Show by an example that not all linear functionals on a normed space are continuous.
2. Let $X = \mathbb{K}^2$ with $\|.\|_1$ and $Y = \{(x(1), x(2)) \in X : x(2) = 0\}$. Find all Hahn-Banach extensions of the continuous linear functional g on Y defined by $g(x(1), x(2)) = x(1)$.
3. Let X be a normed space and E be a subset of X such that f(E) is bounded in $\mathbb{K}$ for every $f \in X'$, the dual space of X. Prove that E is bounded in X.
4. Let X, Y, Z be metric spaces. Suppose $F : X \to Y$ and $G : Y \to Z$ are closed maps. Prove that $G_oF : X \to Z$ is closed.
5. Let X be an inner product space with the inner product $\langle , \rangle$. For all $x, y \in X$, prove that $|\langle x,y \rangle|^2 \leq \langle x,x \rangle \; \langle y,y \rangle$.
6. Let $(x_1, x_2, \dots, x_n)$ be an orthogonal set in an inner product space X and $k_1, k_2, \dots, k_n$ be scalars having absolute value 1. Then prove that $\| kx_1 + \dots + k_nx_n\|^2 = \|x_1\|^2 + \dots + ---- + \|x_n\|^2$.

P.T.O.

PART – B

Unit – I

7. a) If $m(E) = \infty$, where E is a subset of the real line IR, prove that there may be no inclusion relation among the spaces $L^p(E)$.

b) Let X and Y be normed spaces and $X \neq \{0\}$. Prove that BL (X, Y) is a Banach space in the operator norm if and only if Y is Banach.

8. a) Let X and Y be normed spaces and $F \in BL(X, Y)$. Prove that the following statements are equivalent.

i) $\| F \| = \sup \{\| F(x) \| : x \in X, \| x \| \leq 1\}$

ii) $\| F \| = \sup \{\|F(x)\| : x \in X, \|x\| = 1\}$

iii) $\| F \| = \inf \{\alpha \geq 0 : \|F(x)\| \leq \alpha \|x\| \text{ for all } x \in X\}$

b) State and prove Hahn-Banach separation theorem.

9. a) Let Y be a closed subspace of a normed space X. For x + Y in the quotient space X/Y, let $|||x+y||| = \inf \{\| x+y \| : y \in Y\}$.

Prove that $|||\cdot|||$ is a norm on X/Y.

b) Define a Banach space and prove that a Banach space cannot have a denumerable Hamel basis.

Unit – II

10. a) Let X be a Banach space, Y be a normed space and (F_n) be a sequence in BL (X, Y) such that the sequence $(F_n(x))$ converges in Y for every $x \in X$. If $F(x) = \lim_{n \to \infty} F_n(x)$, for $x \in X$, prove that (F_n) converges uniformly to F on every totally bounded subset of X.

b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is continuous if $g \circ F$ is continuous for every $g \in Y'$.

c) Prove that a continuous map from a metric space to a metric space is closed.

11. a) Show by an example a set of continuous functions from a metric space to a metric space can be bounded at each point without being uniformly bounded.

b) State and prove the uniform boundedness principle.

12. a) If F is a closed bijective map from a metric space X onto a metric space Y then prove that F^{-1} is closed.

b) State and prove the open mapping theorem.

Unit – III

13. a) Prove that every orthogonal subset of an inner product space is linearly independent.

b) Let $\{x_1, x_2 ..\}$ be a linearly independent subset of an inner product space X. Prove that there exists an orthonormal set $\{u_1, u_2 ...\}$ in X such that span $\{u_1, u_2, ..., u_n\}$ = span $\{x_1, x_2, .., x_n\}$

c) If $\{x_1, x_2 ..\}$ is a countable orthonormal set in a Hilbert space H and $k_1, k_2....$ is a sequence of scalars such that $\sum_{n=1}^{\infty} |k_n|^2 < \infty$ then prove that $\sum_{n=1}^{\infty} k_n x_n$ converges in H.

14. a) Let F be a subspace of an inner product space X and $x \in X$. Prove that an element $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$.

b) Let H' be the dual space of the Hilbert space H, for $f \in H'$ τf denotes the representor of f in H. If for $f, g \in H'$ $\langle f, g \rangle' = \langle \tau f, \tau g \rangle'$ then prove that $\langle \, , \, \rangle'$ is an inner product on H'. Also prove that H' is a Hilbert space with respect to the inner product $\langle \, , \, \rangle'$.

15. a) Let $\{u_1, u_2, ..\}$ be a countable orthonormal set in an inner product space X and $x \in X$. Then prove that $\sum_n |\langle x, u_n \rangle|^2 \leq \| x \|^2$ ..... (1) and prove that equality holds in (1) if and only if $x = \sum_n \langle x, u_n \rangle u_n$.

b) State and prove projection theorem for Hilbert spaces.
