Unit – III

13. a) Let $E \subset \mathbb{R}^n$ and f a real valued differentiable function with domain E. For $x \in E$ and for a unit vector u in $\mathbb{R}^n$ define the direction derivative $(D_u f)x$ and prove that $(D_u f)(x) = \sum_{i=1}^{n} (D_i f)(x)\, u_i$, where $u_1, u_1 ...... u_n$ are the co-ordinates of u.

b) Let X be a metric space with metric d. If ϕ maps X into X, if X is complete and ϕ is a contraction then prove there exists a unique $x \in X$ such that $\phi(x) = x$.

14. Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable in E. If and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m$ and $1 \le j \le n$.

15. State and prove the inverse function theorem.

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Reg. No. : ....

Name : ....

Third Semester M.A./M.Sc./M.Com. Degree (Reg./Supple./Imp.) Examination, November 2016
MATHEMATICS
MAT 3C14 : Advanced Real Analysis
(2014 Admission Onwards)

Time : 3 Hours Max. Marks : 60

Instructions : *1) Answer* ***four*** *questions from Part –* ***A. Each*** *question carries 3 marks.*

2) Answer ***any four*** *questions from Part –* ***B*** *without omitting* ***any unit. Each*** *question carries* ***12*** *marks.*

PART – A

1. Prove or disprove : 'If a sequence $\{f_n\}$ of differentiable real functions which converges to the differentiable functions f on the interval [a, b] of $\mathbb{R}$ then the sequence $\{f_n'\}$ converges to f' on [a, b]'.

2. For every interval [–a, a] prove that there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim_{n \to \infty} P_n(x) = |x|$.

3. Show that the exponential function e^x defined on $\mathbb{R}' \to +\infty$ as $x \to +\infty$ and $\to 0$ as $x \to -\infty$. Also show that $\lim_{x \to \infty} x^n e^{-x} = 0$ for every n.

4. Show that the function $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$, the space of all complex numbers, is periodic with period $2\pi i$.

5. Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^n$ and $x \in E$. Prove that if the derivative of f at x exists then it is unique.

6. State contraction principle and show that the completeness assumption of the statement of the contraction principle is essential.

PART – B

Unit – I

7. a) For m = 1, 2, ... let $f_m(x) = \lim_{n \to \infty} (\cos m!\pi x)^{2n}$, $x \in \mathbb{R}$

If $f(x) = \lim_{n \to \infty} (\cos m!\pi x)^{2n}$. Then prove that f is bounded on every interval but it is not Riemann-integrable on any closed interval [a, b] of $\mathbb{R}$.

b) Let α be a monotonically increasing function on [a, b]. Suppose $f_n \in R(\alpha)$ on [a, b] for every n. Suppose $f_n \to R$ uniformly on [a, b]. Then prove that $f \in R(\alpha)$ on [a, b]. Also prove that

$$\int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f_n d\alpha .$$

8. a) Prove or disprove 'Every convergent sequence of real continuous functions contains a uniformly convergent subsequence'.

b) If K is a compact metric space, if $f_n \in \mathbb{C}(K)$ for n = 1, 2, ... and if $\{f_n\}$ converges uniformly on K then prove that $\{f_n\}$ is equicontinuous on K.

c) If K is compact, If $f_n \in \mathbb{C}(K)$ for n = 1, 2, ... and if $\{f_n\}$ is point wise bounded and equicontinuous on K then prove that $\{f_n\}$ is uniformly bounded on K.

9. State and prove Weierstrass theorem .

Unit – II

10. a) Let Σc_n be a convergent series of real numbers. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, (–1 < x < 1) then prove that $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$.

b) Suppose the series $\Sigma a_n x^n$ and $\Sigma b_n x^n$ converge in the segment S = (–R, R). Let E be the set of all $x \in S$ at which

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

If E has a limit point in S then prove that $a_n = b_n$, for n = 1, 2, ...

11. a) Let f be Riemann integrable function on [a, b] and $\{\phi_n\}$ a sequence of orthonormal functions on [a, b]. If n is the n^{th} partial sum of the Fourier series of f, and suppose $t_n(x) = \sum_{m=1}^{n} r_m \phi_m(x)$, for some constants r_m's .

Then prove that $\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx$.

b) Define beta function B (x, y) and prove that for x > 0 and y > 0,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

c) Write Stirling's formula.

12. a) Prove that the complex field is algebraically closed.

b) Prove that every continuous periodic functions on $\mathbb{R}$ with period 2π can be approximated by a sequence of trigonometric polynomials.