K15P 0058

Reg. No. : ....

Name : ....

Third Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, November 2015

MATHEMATICS (2014 Admn.)

MAT 3C12 : Functional Analysis

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions from this Part. **Each** question carries **3** marks.

1. Let $n \geq 2$ and $0 < p < 1$. For $x \in K^n$, let $\| x \|_p = \left(\sum_{j=1}^{n} |x(j)|^p \right)^{1/p}$. Determine whether $\| \ \|_p$ is a norms on K^n.

2. Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is continuous if and only if for every Cauchy sequence (x_n) in X, the sequence $(F(x_n))$ is Cauchy in Y.

3. Give an example of a set of continuous functions from a metric space to a metric space that is bounded at each point without being uniformly bounded.

4. Let X, Y and Z be normed spaces. If $F : X \to Y$ is continuous and $G : Y \to Z$ is closed, then show that $G \circ F : X \to Z$ is closed.

5. Let $< , >$ be an inner product on a linear space X and $T : X \to X$ be a one to one linear map. Let $\langle x, y \rangle_T = \langle T(x), T(y) \rangle$; $x, y \in X$. Show that $\langle \ , \ \rangle_T$ is an inner product on X.

6. Let E be a convex subset of an inner product space X. Prove that there exists at most one best approximation from E to any x in X.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **12** marks.

UNIT – I

7. a) Let X be a normed space. Prove that every closed and bounded subset of X is compact if and only if X is finite dimensional. **6**

   b) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range R(F) is finite dimensional. Prove that F is continuous if and only if the zero space Z(F) of F is closed in X. **6**

8. a) State and prove Hahn-Banach extension theorem. **7**

   b) Give an example of non unique Hahn-Banach extension. **4**

   c) State a characterization (without proof) of normed spaces which admit unique Hahn-Banach extensions. **1**

9. a) Prove that a normed space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X. **6**

   b) Prove that a Banach space cannot have a denumerable Hamel basis. **3**

   c) Define a Schauder basis for a normed space and give an example. **3**

UNIT – II

10. a) State and prove uniform boundedness principle. **8**

    b) Show that uniform boundedness principle may not hold if the domain space is not Banach. **4**

11. a) Let X and Y be Banach spaces and $F : X \to Y$ be a closed linear map. Prove that F is continuous. **8**

    b) Let P be a projection on a normed space X such that R(P) and Z(P) are closed in X. Prove that P is a closed map. **4**

12. a) Let X and Y be Banach spaces and let $F \in BL(X, Y)$ be bijective. Prove that $F^{-1} \in BL(Y, X)$. **3**

    b) If $k_n \in K$, $n = 0, \pm 1, \pm 2, \ldots$ such that $k_n \to 0$ as $n \to \pm\infty$, does there exist some $x \in L^{1}([-\pi, \pi])$ such that $\hat{x}(n) = k_n$ for each n ? Justify your answer. **9**

UNIT – III

13. a) State and prove Schwarz inequality on an inner product space. **6**

    b) Prove that an $\langle\, , \,\rangle$ on a linear space X induces a norm on X. **3**

    c) Among all the l^p spaces $1 \le p < \infty$ show that only l^2 is an inner product space. **3**

14. a) Let H be a nonzero Hilbert space over K. Prove that H has a countable orthonormal basis if and only if H is separable. **6**

    b) Let E be a nonempty closed convex subset of a Hilbert space H. For each x in H, prove that there exists a unique best approximation from E to x. **6**

15. a) State and prove projection theorem. **6**

    b) Let (x_n) be a sequence in a Hilbert space H. If (x_n) is bounded, then prove that it has a weak convergent subsequence. **6**
